

Income and inequality under asymptotically full automation

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Abstract

Recent progress in artificial intelligence raises the prospect that, asymptotically, all tasks will be automated. We characterize the consequences for capital and labor markets of such automation when capital returns and wages are determined by standard economic forces, and in the presence of heterogeneous financial frictions. As automation shifts income from labor to capital, savings frictions drive inequality, both directly through wealth accumulation and indirectly through interaction with labor choice. Heterogeneity in financial frictions endogenously creates distinct populations of capitalists and workers. The interaction of savings frictions with labor supply choice can create multiplicity and trap workers in a higher-inequality steady state.

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Technological advance has allowed the automation of many tasks historically performed by workers, and conversely, expanded the set of tasks for which capital is useful. Recent progress in artificial intelligence raises the prospect that, asymptotically, *all* tasks will be automated. Asymptotically full automation in turn raises the prospect that capital markets will grow to dominate the economy, with household income stemming largely from capital holdings. In this eventuality, heterogeneity in households’ ability to hold and manage capital becomes even more important than it already is ([Campbell and Ramadorai, 2025](#)).¹

In this paper, we analyze how heterogeneity in households’ capital-holding abilities affects how full automation plays out. We emphasize three points. First, such heterogeneity allows us to study the distributional consequences that stem from an automation-induced rise in the economy’s capital share. Second, the evolution of the capital share under automation depends on the evolution of labor and capital supply, both of which in turn depend on households’ ability to hold and manage capital. The equilibrium effects that arise prompt questions such as: if some households are worse at holding capital, do they respond by supplying more or less labor, and in turn, does this make *capital dominance* of the economy—a capital share that converges to one—more or less likely? We characterize conditions under which heterogeneity in capital-holding ability not only worsens inequality if capital dominance occurs, but also makes capital dominance more likely to occur in the first place. Third, we use our framework to consider policy responses, including taxation of capital and/or labor income, and “Luddite” policies that directly retard automation. Overall, our paper argues that if automation makes capital central to the economy, then heterogeneity in the capital-holding abilities likewise moves to the center-stage.

The building block for the production-side half of our analysis is [Aghion et al. \(2019, henceforth “AJJ”\)](#). These authors consider an economy with exogenous labor supply and savings decisions in which automated and non-automated tasks are complements, and in which all tasks are asymptotically automated. In this setting, they analyze what standard economic forces imply for prices—here, the rental rate of capital and wages—and in particular establish that capital dominance emerges only if the pace of automation is sufficiently fast relative to the exogenous rate of capital accumula-

¹Similarly, and unsurprisingly, BlackRock Chairman Larry Fink warns: “[T]he vast majority of wealth has flowed to people who owned assets, not to people who earned most of their money by working. Now AI threatens to repeat that pattern at an even larger scale.” ([2026 Annual Letter to Investors](#))

tion. If instead capital accumulation is fast, the expanded supply of capital causes the rental rate to fall by enough to keep the capital share of the economy bounded away from 100%. Looking ahead, an important driver of our results is the parallel but omitted observation that if the labor supply shrinks faster than wages increase faster and capital dominance is again less likely. Taking AJJ’s specification of the production-side as given, we add a household-side to the analysis to capture savings and labor supply decisions, and introduce heterogeneity in households’ ability to hold and manage capital. A key element of our model is complementarity between consumption and leisure, in addition to the aforementioned complementarity between automated and non-automated tasks.²

A first implication to emerge from our analysis is that this simple form of heterogeneity is, by itself, enough to generate stratification across households, with some households resembling “capitalists” and others resembling “workers.” Specifically: in equilibrium, once automation has progressed enough then either some households rely entirely on capital income, or some households rely entirely on labor income. For use throughout, we refer to households that invest effectively as *capitalists*, and other households as *workers*; we stress, however, that there is nothing stopping capitalists from working (or workers from holding capital, albeit at a lower rate of return).

If the pace of automation falls below some threshold—which we characterize and relate to observables—then even as all tasks are asymptotically automated the economy continues to resemble its current form. Labor supply falls gradually (a continuation of already-observed trends); and the combination of (relatively) fast capital accumulation with shrinking labor supply leads to wage growth that is fast enough for the labor share to remain bounded away from zero. Workers don’t hold capital, and economic outcomes are locally independent of workers’ ability to manage capital.

In contrast, workers’ capital-holding abilities play an important role if the rate of automation is higher. The starkest case emerges if the automation speed is intermediate, and complementarities across tasks and between consumption and leisure are both strong. In this case, two asymptotic steady states coexist. One is as just described. But in the other steady state, workers supply

²Complementarity across tasks implies that the automation of some tasks increases the marginal product of labor in yet-to-be-automated tasks—a central insight of [Baumol and Bowen \(1965\)](#). Consumption-leisure complementarity helps to generate dynamics consistent with time-series and cross-country evidence (e.g., [Becker, 1965](#); [Huberman and Minns, 2007](#); [Feenstra et al., 2015](#)). [Keynes \(1930\)](#) famously predicted a 15-hour workweek for his grandchildren thanks to rising productivity. [Boppart and Krusell \(2020\)](#) write: “As it turned out, Keynes was wildly off quantitatively, but he was right qualitatively (on this issue).” [Bick et al. \(2018\)](#) further find that for most countries, hours worked are decreasing in the wage.

more labor, pushing both wages and consumption down, in turn incentivizing workers to supply more labor via consumption-leisure complementarity. Strong complementarity across automated and non-automated tasks generates a large reduction in both wages and labor income. Workers respond by saving more, even despite their disadvantage in holding capital. Importantly, the worse workers are at holding capital, the wider is the range of parameters for which a steady state of this type exists, because a key element supporting it is workers' low level of consumption.

For high rates of automation, the economy converges to capital dominance. Here, heterogeneity in the ability to hold and manage capital translates one-for-one into heterogeneity in consumption growth and workers face (relative) immiseration. This prospect echoes the current public debate about possible policy responses, in particular focusing on capital-tax-funded handouts and the deliberate retardation of automation.³ Both policies change the relative rate of capital accumulation and automation, and both lower the overall growth rate. Our analysis speaks naturally to the efficacy of these policies, and implies that the taxation and redistribution of capital income is the better policy. In the same spirit, labor regulations that cap hours worked (potentially motivated as a way to share a shrinking pool of available jobs among working households) are likewise dominated by capital-taxation-and-redistribution. For the multiplicity case, the tax-and-redistribute policy can rule out the capital-dominant steady state, with zero tax revenue collected in equilibrium.

Our model delivers analytical expressions for the thresholds between the above regimes, and these are simple enough to relate them to observables. A first-pass calibration of the fully-characterized capital-dominance threshold suggests that the current speed of automation in the US is too slow for capital dominance (with or without multiplicity), and accordingly, that the current path of automation may not result in the kind of profound changes associated with capital dominance. Instead, and to a perhaps surprising extent, the economy may continue to resemble its current form, even as full automation is asymptotically approached. But the existence of threshold rates also highlights why the prospect of faster automation should be taken seriously in the age of AI.

³For example, [Brookings](#) (Jul 3, 2024) reports “[A]bout half of Americans think that the increased use of AI will lead to greater income inequality and a more polarized society. Roughly two thirds think the government should take action to prevent the loss of jobs due to AI.” Bill Gates adds: “The robot that takes your job should pay taxes.” ([Quartz](#), Feb 17, 2017). Elon Musk comments: “There is a pretty good chance we end up with a universal basic income, or something like that, due to automation” ([CNBC](#), Nov 4, 2016); and Sam Altman “[expects] ... some change required to the social contract given how powerful we expect this technology to be.” ([AI for Good Summit](#), May 30, 2024). See also, e.g., [Costinot and Werning \(2022\)](#).

For the case in which the speed of automation does accelerate by enough to trigger capital dominance, one possibility is that automation will cure its own ills by narrowing the investment-return wedge between capitalists and workers. But even in this case, the return wedge is likely to disappear only gradually, with worker-capitalist inequality growing in the meantime. The calibrated version of our model casts quantitative light on this, and suggests a substantial rise in worker-capitalist inequality even in this (relatively rosy) scenario (see 5.1).

Our aim in this paper is to take seriously the prospect that all tasks will eventually be automated, and to analyze the consequences for the economy. As such, we take the speed of automation as exogenous; the key endogenous objects are capitalists’ and workers’ savings and labor responses, and the associated equilibrium capital return rates and wages. In subsections 4.1.4 and 5.2 we briefly describe the likely consequences of endogenous automation.⁴ More generally, we deliberately study the effects of asymptotic automation in a standard model with minimal frictions; our main interest is to characterize how asymptotic automation plays out in the context of widely accepted economic forces and in the presence of widely documented heterogeneity in financial outcomes.

Related literature. Our model speaks to two broad strands of literature in finance and macroeconomics—the household finance literature on asset market participation and portfolio allocation, and the macro literature on automation, factor shares, and, more recently, artificial intelligence.

The household finance literature documents heterogeneity across households in investment decisions and returns, and the distributional consequences (e.g., [Campbell, 2006](#); [Calvet et al., 2007](#); [Gomez, 2024](#)). We take this vast empirical evidence of heterogeneity and explore the interaction with a first-order consequence of automation as a shifter of factor shares. We model this heterogeneity as a wedge in investment returns, but note that this should be interpreted broadly. One interpretation is that households differ in their investment abilities. Empirically, high-income households earn higher risk-adjusted returns (e.g., [Fagereng et al., 2020](#); [Bach et al., 2020](#); [Smith et al., 2023](#)), meaning this channel is applicable to our model (which abstracts from risk). Moreover, dif-

⁴[Acemoglu and Restrepo \(2018b\)](#) endogenize automation and the invention of new tasks in which labor has a comparative advantage. They find that long-run factor shares are stable if the long-run rental rate of capital is sufficiently high. Intuitively, automation reduces the cost of labor, thereby discouraging further automation and encouraging the development of new tasks. In our case, the labor share stabilizes despite exogenous automation of all tasks in the limit. Our mechanism works through complementarity across tasks (the Baumol effect) and between consumption and leisure. The two channels operate independently.

ferences in investment ability are mathematically isomorphic to differences in time preference rates (e.g., [Lawrance, 1991](#); [Epper et al., 2020](#)), which are often associated with wealth inequality (e.g., [Ramsey, 1928](#); [Krusell and Smith, 1998](#)). Heterogeneity in risk aversion, background risk, or ability to diversify would generate similar effects. For evidence on heterogeneity in time preference and risk aversion, see [Cronqvist and Siegel \(2015\)](#) and references therein.

Since our framework does not distinguish between investments in asset markets and direct investment of capital within a non-financial firm, heterogeneity could further stem from differences in entrepreneurial skill. As for investment decisions, the distributional consequences of entrepreneurship are often tied to financial frictions ([Hurst and Lusardi, 2004](#); [Cagetti and De Nardi, 2006](#); [Irie, 2024](#)). Another interpretation is through intermediation rents paid by some households and earned by others ([Philippon, 2015](#)), although the effects we document arise regardless of whether the wedge is a deadweight cost or earned by capitalists. We assume that this heterogeneity is persistent, and, with the exception of [Section 5.1](#), unaffected by the technological progress that drives automation.⁵

Similarly to our paper, [Moll et al. \(2022\)](#) combine investment decisions with automation. They show that automation increases inequality via returns to wealth and by facilitating stagnant wages. As in their model, returns to capital rise with the speed of automation (up to the capital-dominance threshold), and capital income tends to generate inequality in consumption growth and consumption shares. Unlike in their model, where inequality is driven by idiosyncratic heterogeneity in realized returns, our result is obtained from systematic financial frictions interacting with automation-induced shifts in factor shares. We also find that automation *can* reduce wages, but only does so under specific circumstances. For low-to-medium automation rates, wage growth rises with faster automation.

Our conceptualization of the automation process directly follows the insightful work of [AJJ](#).⁶

⁵For empirical evidence that return heterogeneity across households is persistent, even across generations, see [Fagereng et al. \(2022, 2021\)](#). Moreover, return heterogeneity will survive technological progress if, for instance, the proliferation of AI in financial markets takes the form of an arms race among sophisticated investors, if data and machine learning are used to exploit lack of sophistication ([Campbell and Ramadorai, 2025](#)), or if the adoption of highly sophisticated robo-advisors by retail investors is bounded by, for example, behavioral frictions (see [Greig et al. \(2024\)](#) for evidence). See also [Section 5.1](#).

⁶In work that postdates ours, [Korinek and Suh \(2024\)](#) similarly endogenize savings, but focus instead on the importance of full automation in finite time vs asymptotic full automation. Unlike us, they study a representative agent economy with exogenous labor supply. [Guerreiro et al. \(2022\)](#) and [Ray and Mookherjee \(2022\)](#) study settings in which all routine tasks can be automated immediately, and only the cost of automation prevents this from happening. Instead, a crucial assumption for the Baumol-force to operate is that at any finite time some tasks cannot be automated, though the number of such tasks asymptotes to zero.

Following earlier models of automation, we view technological progress as a gradual replacement of labor with capital in production (Zeira, 1998; Acemoglu and Restrepo, 2018a). On this view, automation extends beyond AI to major sources of economic growth since the Industrial Revolution.

Our formulation of the capital-labor complementarity is distinct from the literature explaining changes in skill premia through skill-biased technological change, that is, capital-skill complementarity in production (e.g., Acemoglu, 1998; Krusell et al., 2000; Autor et al., 2003). Instead, we do not take a stance on the types of tasks that remain unautomated for longer, meaning the wages from those tasks may be earned by nurses, teachers, or—as Baumol and Bowen (1965) would have it—performing artists. Acemoglu (2024) finds that the impact of AI on productivity is more equally distributed across demographic groups than that of earlier forms of automation, and consequently predicts a smaller effect on inequality. Like the literature on skill-biased technological change, this argument focuses on intra-labor inequality. Acemoglu (2024) also estimates the likely AI-driven automation rate over the next ten years and we use these estimates in our calibration. Instead, we note that, like other forms of automation, AI shifts income from labor to capital, and we model the distributional effects of this shift in a world with heterogeneous investment returns, providing a natural household-finance analog to the literature on labor skill bias.

Trammell and Korinek (2023) synthesize the literature on AI-driven automation. They emphasize the importance of considering frameworks in which the long-run equilibrium differs from the “Kaldor Facts that described growth over the past century or two.” Like Jones and Liu (2024), our model features capital-embodied technical change that is labor-augmenting in the aggregate and able to generate balanced growth. Relative to Jones and Liu (2024), we endogenize labor supply and show that automation can facilitate balanced growth in capital, output, and consumption while, thanks to consumption-leisure complementarity, generating the observed decline in hours worked.

Our analysis is predominantly concerned with the limit of full automation and the asymptotic factor shares. Nonetheless, the key forces driving our mechanism also speak to three long-run empirical trends: (i) the decline in hours worked (Boppart and Krusell, 2020), (ii) the falling labor share (Karabarounis and Neiman, 2014; Barkai, 2020), which our model naturally connects to a decline in TFP growth (Philippon, 2023), and (iii) a reallocation in output shares towards services (Boppart, 2014). We discuss these trends in the context of the model in Subsection 5.4.

1 Model

1.1 Preferences and endowments

There is a unit mass of infinitely lived economic agents, each of whom continuously consumes, works, and adjusts capital holdings. Population growth equals 0 (we have verified that, as in the standard neoclassical growth model, steady state outcomes are independent of population growth). Each agent discounts the future at rate ρ . There is no uncertainty.

Agents are either “workers” or “capitalists” (denoted by subscript ‘o’ for “owners”) with respective measures λ_w and λ_o . The only difference between the two groups is that capitalists are more effective at holding capital. Let $K_{i,t}$, $L_{i,t}$, and $C_{i,t}$ respectively denote the date t capital holding, time spent working, and consumption of an agent of type $i = w, o$. Moreover, let W_t and R_t denote the date t wage rate and return on capital (not including depreciation and other holding costs). Capital accumulation for a type- i agent is

$$\dot{K}_{i,t} = R_t K_{i,t} + W_t L_{i,t} - \delta_i K_{i,t} - C_{i,t}, \quad (1)$$

where δ_i is the combined depreciation and holding costs experienced by type- i agents, where⁷

$$\delta_w > \delta_o.$$

One interpretation of this crucial assumption is that workers make investment mistakes, e.g., due to inertia or lack of awareness. This behavioral interpretation is closely related to the observation that a difference in δ is mathematically isomorphic to a spread in time preference rates, ρ .

Importantly, we assume that the frictions that give rise to the difference in δ s cannot be overcome by intermediation, e.g., with capitalists investing frictionlessly on behalf of workers. In this spirit, a second interpretation is that financiers extract intermediation rents from workers. We do not model financiers explicitly, meaning that these rents can take the form of a deadweight cost, or, if capitalists act as financiers, a transfer from workers to capitalists. In the limits we study, these turn out to be equivalent. To see this, denote by $\iota > 0$ the rate of intermediation rent paid on worker

⁷See Appendix C for an analysis of the (easier) representative agent case.

capital. Then write $\delta_w = \delta + \iota$ and $\delta_o = \delta - \iota \frac{K_w}{K_o}$ to reflect the expense of workers and, respectively, the financier income of capitalists. To capitalists, this income is independent of their capital choice, but expressed relative to K_o to compress notation into the depreciation rate. In all equilibria that we study $\lim_{t \rightarrow \infty} \frac{K_w}{K_o} = 0$, and so this interpretation is equivalent to one in which workers pay a deadweight cost that raises their depreciation rate above that of capitalists.

Each agent's flow endowment of time is 1, so that flow leisure is $1 - L_{i,t}$. Regardless of type, each agent's flow utility is

$$\frac{1}{1-\gamma} \left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega (1 - L_{i,t})^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\gamma}{1-\frac{1}{\eta}}}.$$

Here, ω parameterizes the relative importance of leisure and consumption; η is the elasticity of substitution between consumption and leisure; while γ is the standard coefficient from power utility functions. In the special case of $\omega = 0$, the intertemporal elasticity of substitution is $\frac{1}{\gamma}$.

Capital holdings must be non-negative, and the standard transversality condition must hold:

$$K_{i,t} \geq 0 \tag{2}$$

$$\lim_{t \rightarrow \infty} K_{i,t} \int_0^t e^{-(R_s - \delta_i)} ds = 0. \tag{3}$$

1.2 Technology

The production-side of the economy matches AJJ (see introduction). Specifically, output is a composite consumption good composed of a unit measure of complementary “tasks,” with a constant elasticity of substitution σ across any pair of tasks. A “task” should be interpreted generally. In contrast to [Acemoglu and Restrepo \(2018b\)](#), we think of tasks as being fundamental “needs” such as food, shelter, entertainment, transport etc., so that the set of tasks is fixed over time (as in AJJ).

Importantly, tasks are gross complements, i.e., $\sigma < 1$. It is this assumption that allows the Baumol force to potentially operate. The interpretation of $\sigma < 1$ nests both preference-based complementarity across different consumption goods and technology-based complementarity in production processes that combine intermediate tasks into ultimate output goods.

Let α_t be the fraction of tasks that has been automated at date t . Non-automated tasks are executed using only labor. For automated tasks, capital and labor are perfect substitutes. In equi-

librium, capital grows without bound, so capital becomes abundant relative to labor; consequently, in equilibrium automated tasks are (eventually) executed using only capital.

Let K_t and L_t denote aggregate capital and labor, i.e., $K_t = \lambda_w K_{w,t} + \lambda_o K_{o,t}$ and $L_t = \lambda_w L_{w,t} + \lambda_o L_{o,t}$. Hence date t output is

$$\begin{aligned} Y_t = F(K_t, L_t; \alpha_t) &= \left(\alpha_t \left(A_K \frac{K_t}{\alpha_t} \right)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_t) \left(A_L \frac{L_t}{1 - \alpha_t} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \\ &= \left(\alpha_t^{\frac{1}{\sigma}} (A_K K_t)^{\frac{\sigma-1}{\sigma}} + (1 - \alpha_t)^{\frac{1}{\sigma}} (A_L L_t)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}. \end{aligned} \quad (4)$$

That is: each of the α_t automated tasks receives capital $\frac{K_t}{\alpha_t}$, and each of the $1 - \alpha_t$ non-automated tasks receives labor $\frac{L_t}{1 - \alpha_t}$. Parameters A_K and A_L determine the productivity of capital and labor; allowing for labor-augmenting technological change ($g_{A_L} \neq 0$) is straightforward (Section 6).

For calibration (Section 6), note that the elasticity of substitution across tasks, σ , coincides with the elasticity of substitution between capital and labor, which is estimated by a sizeable literature.

The marginal products of capital and labor are

$$F_{K,t} = \alpha_t^{\frac{1}{\sigma}} A_K^{\frac{\sigma-1}{\sigma}} K_t^{-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}} \text{ and } F_{L,t} = (1 - \alpha_t)^{\frac{1}{\sigma}} A_L^{\frac{\sigma-1}{\sigma}} L_t^{-\frac{1}{\sigma}} Y_t^{\frac{1}{\sigma}}. \quad (5)$$

As time passes, more and more tasks are automated. Our focus is on the consequences of automation, so we take the automation process as exogenous, reflecting an inexorable “march of progress.” Every instant, a constant share, $\theta > 0$, of previously unautomated tasks gets automated:

$$\dot{\alpha}_t = (1 - \alpha_t) \theta.$$

While we do not model individual firms, this march of progress is consistent with both within-firm automation (Hubmer and Restrepo, 2024) and rising capital intensity among newly entering firms (Barkai and Panageas, 2024). Asymptotically all tasks are automated; but at any finite time t , some tasks remain non-automated.

Looking ahead, faster automation is associated with lower relative wages in equilibrium, i.e., lower $\frac{F_{L,t}}{F_{K,t}}$. Consequently, the likely effects of endogenizing the pace of automation hinge on whether

innovation is labor- or capital-intensive. Labor-intensive automation would amplify the effects of exogenous variation in automation rates. In contrast, capital-intensive automation—often invoked in “singularity” discussions—would dampen the effects of exogenous variation. Section 6 also discusses how labor-augmenting technological change (A_L growing over time) would affect our results.

1.3 Equilibrium

An equilibrium consists of paths $\{K_{i,t}, C_{i,t}, L_{i,t}\}$ for $i = w, o$ and rental rates and wages $\{R_t, W_t\}$ such that $\{K_{i,t}, C_{i,t}, L_{i,t}\}$ is individually optimal for each agent given the path of $\{R_t, W_t\}$, while rental rates and wages are determined by the competitive conditions

$$R_t = F_K(K_t, L_t; \alpha_t) \tag{6}$$

$$W_t = F_L(K_t, L_t; \alpha_t). \tag{7}$$

1.4 Parameter assumptions

To capture the Baumol effect, and consistent with empirical estimates, tasks are gross complements:

$$\sigma < 1.$$

We further assume that consumption and leisure are gross complements,

$$\eta < 1.$$

Under the alternative assumption ($\eta > 1$), many parameter configurations deliver equilibria in which leisure converges to 0, while observed trends indicate increases in leisure. These trends also motivate our deviation from KPR-preferences (King et al., 1988) which generate stable labor supply. Appendix D analyzes the case of $\eta = 1$, in which preferences are Cobb-Douglas and, thus, KPR. Our main results continue to hold but hours worked do not decline.

Throughout, we assume that for $i = o, w$

$$A_K - \delta_i > \rho > (1 - \gamma)(A_K - \delta_i). \quad (8)$$

The first inequality ensures capital growth in a benchmark economy with production $A_K K_t$, while the second inequality ensures the transversality condition is satisfied in the same benchmark.

1.5 Labor share dynamics and capital dominance

We first define our notions of a stable labor share and capital dominance; note some important implications of these definitions; and derive laws of motion for key aggregate quantities. This section uses only the definition of the production function (4).

The date- t labor share of the economy is, as usual, $X_t \equiv \frac{L_t F_{L,t}}{Y_t} = 1 - \frac{K_t F_{K,t}}{Y_t}$. From (4), the ratio of the capital share to the labor share is

$$\frac{1 - X_t}{X_t} = \left(\frac{\alpha_t}{1 - \alpha_t} \right)^{\frac{1}{\sigma}} \left(\frac{K_t}{L_t} \right)^{\frac{\sigma-1}{\sigma}}. \quad (9)$$

The above illustrates the Baumol force highlighted by AJJ. A ceteris-paribus rise in the share of automated tasks, α , reduces the labor share. However, as capital accumulates relative to labor, automated tasks become relatively abundant, and task-complementarity ($\sigma < 1$) ensures that the price effect outweighs the quantity effect, thus exerting upward pressure on the labor share.

Definition 1 *We say that capital dominance occurs if $\lim_{t \rightarrow \infty} X_t = 0$. If instead $\lim_{t \rightarrow \infty} X_t > 0$, we say that the labor share is stable.*

Throughout, we write \lim for $\lim_{t \rightarrow \infty}$, and typically omit time subscripts when characterizing asymptotic behavior.

Let g_R and g_W denote the growth rates of return-on-capital R_t and wages W_t , with parallel notation for growth rates of other quantities. From (5),

$$g_{R,t} = \frac{1}{\sigma} \left(g_{Y,t} - g_{K,t} + \theta \frac{1 - \alpha_t}{\alpha_t} \right), \quad (10)$$

$$g_{W,t} = \frac{1}{\sigma} (g_{Y,t} - g_{L,t} - \theta), \quad (11)$$

Hence capital and labor shares $1 - X$ and X evolve according to

$$g_{1-X,t} = g_{K,t} + g_{R,t} - g_{Y,t} = (1 - \sigma) g_{R,t} + \theta \frac{1 - \alpha_t}{\alpha_t}. \quad (12)$$

$$g_{X,t} = g_{L,t} + g_{W,t} - g_{Y,t} = (1 - \sigma) g_{W,t} - \theta. \quad (13)$$

So capital dominance occurs if $\lim g_W < \frac{\theta}{1 - \sigma}$, while a stable labor share requires

$$\lim g_W = \frac{\theta}{1 - \sigma}. \quad (14)$$

Lemma 1 *Output evolves according to*

$$g_{Y,t} = (1 - X_t) g_{K,t} + X_t g_{L,t} + \frac{\theta}{1 - \sigma} \left(1 - \frac{1 - X_t}{\alpha_t} \right). \quad (15)$$

We characterize the economy as the fraction of automated tasks α_t approaches 100%. We focus on equilibria in which the capital share has a well-defined and strictly positive limit. From (10) and (12) it is immediate that, asymptotically, output and capital grow at the same rate:

$$\lim g_Y = \lim g_K. \quad (16)$$

From (10), the rental rate $F_{K,t}$ asymptotically converges; define $\bar{F}_K \equiv \lim F_K$. Noting that the capital share is straightforwardly a function of the rental rate,

$$1 - X_t = \alpha_t \left(\frac{F_{K,t}}{A_K} \right)^{1 - \sigma}, \quad (17)$$

it follows that in a capital-dominant equilibrium,

$$\lim \frac{Y}{K} = \bar{F}_K = A_K. \quad (18)$$

Finally, the bounded nature of labor L_i and leisure $1 - L_i$ means that, provided labor has a well-defined limit, the asymptotic growth rates of both leisure and labor are weakly negative, i.e., $\lim g_{1-L} \leq 0$ and $\lim g_L \leq 0$, with at least one of these inequalities holding with equality.

2 Equilibrium characterization

2.1 Optimality conditions

The marginal utilities of consumption and leisure are

$$MU_{C_{i,t}} = C_{i,t}^{-\frac{1}{\eta}} \left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega (1 - L_{i,t})^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\eta\gamma}{\eta-1}} \quad (19)$$

$$MU_{1-L_{i,t}} = \omega (1 - L_{i,t})^{-\frac{1}{\eta}} \left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega (1 - L_{i,t})^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\eta\gamma}{\eta-1}}. \quad (20)$$

The intratemporal and intertemporal optimality conditions are

$$W_t C_{i,t}^{-\frac{1}{\eta}} \leq \omega (1 - L_{i,t})^{-\frac{1}{\eta}}, \quad (21)$$

$$\frac{\partial}{\partial t} \ln MU_{C_{i,t}} \leq -(R_t - \delta_i - \rho), \quad (22)$$

with equality if, respectively, labor and capital-holding are strictly positive ($L_{i,t} > 0$, $K_{i,t} > 0$).

Looking ahead: Heterogeneity in δ_i makes the corners of no-work and no-capital relevant.

If type- i agents work then, from (21),

$$g_{C_i} - g_{1-L_i} = \eta g_W \quad (23)$$

It follows that if type- i agents work then their marginal utility grows according to

$$\frac{\partial}{\partial t} \ln MU_{C_i} = -\gamma g_{C_i} - (1 - \eta\gamma) \frac{\omega^\eta g_W}{W^{\eta-1} + \omega^\eta}, \quad (24)$$

while if they are at the no-work corner,

$$\frac{\partial}{\partial t} \ln MU_{C_i} = -\frac{1}{\eta} g_{C_i} + \frac{1 - \eta\gamma}{\eta} \frac{g_{C_i} C_i^{\frac{\eta-1}{\eta}}}{C_i^{\frac{\eta-1}{\eta}} + \omega} = -g_{C_i} \frac{\gamma C_i^{\frac{\eta-1}{\eta}} + \frac{\omega}{\eta}}{C_i^{\frac{\eta-1}{\eta}} + \omega}. \quad (25)$$

2.2 Equilibrium, and the role of the transversality condition (3)

A first step to solving for the equilibrium of the economy is to note that it is enough to keep track of quantities $(K_{o,t}, K_{w,t}, C_{o,t}, C_{w,t})$, because at each instant t these quantities uniquely determine the rental rate and wage R_t and W_t , and labor supply decisions $L_{o,t}$ and $L_{w,t}$.

Lemma 2 *At any time t and for any $(K_{o,t}, K_{w,t}, C_{o,t}, C_{w,t}, \alpha_t)$, there are unique values of $R_t, W_t, L_{o,t}$, and $L_{w,t}$ that satisfy the equilibrium conditions (6), (7), and (21) for $i = o, w$.*

Lemma 2 in turn implies:

Lemma 3 *There exists a function $H : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ such that a path $(K_{o,t}, K_{w,t}, C_{o,t}, C_{w,t})$ is an equilibrium if and only if it satisfies the ODE*

$$\left(\dot{K}_{o,t}, \dot{K}_{w,t}, \dot{C}_{o,t}, \dot{C}_{w,t} \right) = H(K_{o,t}, K_{w,t}, C_{o,t}, C_{w,t}, \alpha_t), \quad (26)$$

given initial capital holdings $K_{o,t=0}, K_{w,t=0}$ and the transversality conditions (3) ($i = o, w$).

The part of Lemma 3 relating to capital K_o and K_w is immediate from the law of motion for capital (1) and Lemma 2. The part of Lemma 3 relating to consumption C_o and C_w follows from the fact that the law of motion governing wages (7) and the intratemporal condition (23) can be solved to give the evolution of wages $g_{W,t}$ and labor $g_{L_{o,t}}, g_{L_{w,t}}$ as functions of capital and consumption growth, which can then be substituted into (22) to determine the evolution of consumption.

The value of Lemma 3 is that it clarifies the relevant boundary conditions. Solving the ODE requires four boundary conditions. The initial capital holdings of capitalists and workers provide two. The remaining two boundary conditions then stem from the transversality conditions for capitalists and workers. Specifically, and as characterized below, these conditions imply asymptotic growth rates for capitalists' and workers' capital holdings.

2.3 From capital-holding ability to “workers” and “capitalists”

Recall that the sole source of heterogeneity in the economy is differences in households' ability to hold and manage capital, δ_i . Our first substantive result establishes that this heterogeneity is

enough to generate the endogenous emergence of households that indeed resemble “workers” and “capitalists.” This endogenous stratification in turn allows us to consider both the distributional consequences of automation, and how automation’s effects depend on differences in δ_i .

As a preliminary step, note that the transversality condition (3), the assumption that capital is sufficiently productive to drive long-run growth (8), and the complementarity of consumption and leisure ($\eta < 1$), together imply that in all equilibria:

Lemma 4 *Asymptotically, the leisure growth rate of both groups is 0, $\lim g_{1-L_i} = 0$; and wages and consumption of both groups grow at a strictly positive rate.*

Because asymptotic leisure growth is 0, if both workers and capitalists work asymptotically then their consumption growth rates would coincide, by (23). In this case, it is impossible to satisfy the intertemporal optimality condition (22) with equality for both groups. Consequently:

Corollary 1 *At least one group must be either at the no-capital corner or the no-labor corner.*

By Lemma 4, consumption grows without bound for both groups, as does the wage rate. From (24) and (25), it follows that regardless of whether or not a group $i = o, w$ works

$$\lim \frac{\partial}{\partial t} \ln MU_{C,i} = -\frac{1}{\eta} \lim g_{C_i}, \quad (27)$$

and the asymptotic intertemporal condition is

$$\lim g_{C_i} \geq \eta (\bar{F}_K - \delta_i - \rho), \quad (28)$$

with equality for any group that holds capital.

Our main result in this subsection characterizes which of the no-work and no-capital corners are relevant. It also justifies our terminology of “workers” and “capitalists.”

Proposition 1 *Asymptotically:*

(I) *Capitalists always hold capital and workers always work. In a capital-dominant equilibrium, capitalists do not work. In a stable labor share equilibrium, workers do not hold capital.*

(II) *Capitalists’ consumption grows weakly faster than workers’ consumption, $\lim g_{C_o} \geq \lim g_{C_w}$.*

(III) Asymptotically, output, capitalists' consumption, and capitalists' capital-holdings all grow at the same rate, $\lim g_Y = \lim g_{C_o} = \lim g_{K_o}$.

In addition to establishing that heterogeneity in δ_i delivers segmentation of economic roles, Part (II) confirms the natural conjecture that capitalists' consumption grows at least as fast as workers' consumption; and so in turn (Part (III)) the growth rate of the overall economy is an accurate proxy for capitalists' consumption growth, but not necessarily for workers' consumption growth.

Inequality (2) imposes the natural requirement that capital holdings are non-negative. Proposition 1 raises the question of whether workers would want to borrow from capitalists, effectively obtaining a negative capital position via financial markets. The answer is no. Specifically, workers won't borrow from capitalists provided that

$$\frac{\partial}{\partial t} \ln MU_{C,w} \geq \frac{\partial}{\partial t} \ln MU_{C,o}. \quad (29)$$

It is immediate that the no-borrowing condition (29) holds for t far enough in the future either if both workers and capitalists hold capital, or if both work.⁸ Moreover, (29) also holds in the remaining case in which workers don't hold capital and capitalists don't work; from (23) and (27), the growth rate of workers' marginal utility approaches $-g_W$, while the growth rate of capitalists' marginal utility is lower, reflecting the fact that they choose not to work.

2.4 Capital-dominant equilibria

We next establish conditions for capital dominance to emerge. There are two possibilities, covered in turn by Propositions 2 and 3: either workers hold capital, or else they live hand-to-mouth on their labor income, even as labor income becomes a vanishingly small share of the economy.

Proposition 2 *A capital-dominant equilibrium in which workers hold capital exists if*

$$\theta \geq (1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o). \quad (30)$$

⁸If workers and capitalists both hold capital then intertemporal optimality (22) for each group implies that condition (29) holds strictly. If workers and capitalists both work then expression (24) and the growth rate comparison in Proposition 1(II) directly implies (29).

Consumption growth of group i satisfies

$$\lim g_{C_i} = \eta (A_K - \delta_i - \rho). \quad (31)$$

Labor converges to 0 according to

$$\lim g_{L_w} = (\eta - \sigma) (A_K - \delta_w - \rho) + \eta (\delta_w - \delta_o) - \theta.$$

Proposition 3 *A capital-dominant equilibrium in which workers do not hold capital exists if complementarities are weak,⁹ $\sigma + \eta > 1$, and*

$$\theta \in [(1 - \sigma) (A_K - \delta_o - \rho), (1 - \sigma) (A_K - \delta_w - \rho) + \eta (\delta_w - \delta_o)]. \quad (32)$$

Capitalists' consumption growth satisfies (31), while workers' consumption growth satisfies

$$\lim g_{C_w} = \eta \frac{\lim g_{C_o} - \theta}{\sigma + \eta - 1} < \lim g_{C_o}. \quad (33)$$

Labor converges to 0 according to

$$\lim g_{L_w} = \frac{\eta - 1}{\eta} \lim g_{C_w}. \quad (34)$$

From Propositions 2 and 3, capital dominance emerges when the rate of automation is sufficiently high.¹⁰ In particular, the Baumol effect, arising from task-complementarity ($\sigma < 1$) pushes against capital dominance. Capital accumulation, which is asymptotically proportional to $A_K - \delta_o - \rho$ in a capital-dominant equilibrium, likewise pushes *against* capital dominance because it increases wages relative to the return on capital. Capital dominance emerges when automation advances sufficiently rapidly relative to the extent of complementarity and the rate of capital accumulation.

⁹If complementarities are strong ($\sigma + \eta < 1$) then an unstable equilibrium in which workers do not hold capital exists for automation speeds above the threshold (35) but below the threshold (30). As the proof of Proposition 3 shows, the equilibrium is unstable because a perturbation in hours-worked is self-reinforcing under strong complementarities. See also the discussion in Section 3.1.

¹⁰See footnote 8 of AJJ for a related statement in a representative-agent model with exogenous capital accumulation and labor supply.

As one would expect, capital dominance is associated with an increase in the return on capital:

Corollary 2 *The return on capital in a capital-dominant equilibrium exceeds today's rate.*

Capital dominance is associated with the immiseration of workers *relative* to capitalists. This is immediate if workers do not hold capital (Proposition 3). But even when workers hold capital, they are disadvantaged relative to capitalists in doing so ($\delta_w < \delta_o$). At the same time: Even in a capital-dominant equilibrium workers' consumption grows without bound, even as their leisure approaches its upper bound.

2.5 Stable labor share equilibria

If instead automation proceeds more slowly, a stable labor share emerges. In this case, workers' and capitalists' consumption grow at the same rate.

Proposition 4 *A stable labor share equilibrium exists if*

$$\theta < (1 - \sigma)(A_K - \delta_o - \rho). \quad (35)$$

Consumption of capitalists and workers grows at same rate,

$$\lim g_{C_o} = \lim g_{C_w} = \frac{\eta\theta}{1 - \sigma}. \quad (36)$$

Labor converges to 0 according to (34). The labor share converges to

$$\lim X = 1 - \left(\frac{\delta_o + \rho + \frac{\theta}{1 - \sigma}}{A_K} \right)^{1 - \sigma}. \quad (37)$$

3 Heterogeneity in capital-holding ability

Together, Propositions 2-4 span the parameter space, and are illustrated by Figure 1, which displays the cases of weak and strong complementarities separately. In both cases, whether or not the economy ends up with capital dominance depends on whether the speed of automation θ exceeds

some threshold. To reiterate, the underlying economic mechanism is that capital dominance arises only if automation is sufficiently fast to keep capital scarce relative to labor.¹¹ In the case of strong complementarities, multiple asymptotic steady states coexist for intermediate speeds of automation; see discussion immediately below. In this section, we expand on the ways in which equilibrium outcomes depend on heterogeneity in households' ability to hold and manage capital.

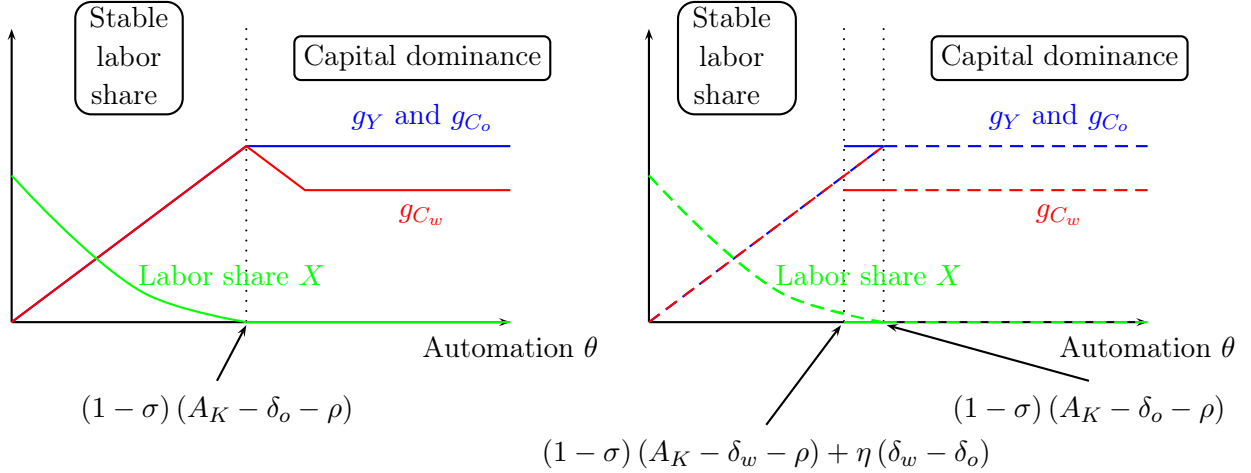


Figure 1: Asymptotic growth rates and labor share as a function of the automation speed θ . The left and right panels show the cases of weak complementarities ($\sigma + \eta > 1$) and strong complementarities ($\sigma + \eta < 1$). Dashed and solid lines correspond to multiple steady states that arise for intermediate automation speeds.

3.1 Multiplicity

When the combined complementarity of tasks (σ) and of consumption and leisure (η) is strong, stable labor share and capital-dominant steady states coexist for some parameters (Propositions 2 and 4). Coexistence arises because capital dominance is associated with lower worker consumption, which is in turn self-reinforcing: complementarity of consumption and leisure implies that lower

¹¹Given the roles of capital accumulation and labor supply, it is noteworthy that the preference parameters γ (the power utility parameter) and η (elasticity of substitution between consumption and leisure) do not feature in the threshold condition (35). The reason is that the IES and consumption-leisure substitutability enter the threshold condition only via their ratio. Further, regardless of the specific equilibrium, consumption grows without bound while leisure approaches its upper bound. It follows that the asymptotic marginal utility of consumption is determined by consumption-leisure substitutability η (specifically, is proportional to $C_i^{-\frac{1}{\eta}}$), and hence the IES converges to consumption-leisure substitutability η . Appendix D analyzes the equilibrium under KPR preferences (King et al., 1988), which arise if $\eta = 1$. In this case, labor doesn't shrink, and the capital-dominance threshold depends on γ and η (see Proposition D-6).

consumption is associated with workers supplying more labor; and this association is especially strong when η is low. Larger labor supply is in turn associated with lower wages. Because task complementarity is strong, the net effect of more labor but lower wages is to significantly reduce labor income—which in turn leads to significantly lower worker consumption.

For a simple illustration of the source of multiplicity, consider a static setting with hand-to-mouth workers. Intratemporal optimality (21) and hand-to-mouth consumption, $C = WL$, imply

$$(1 - L)^{-\frac{1}{\eta}} = \omega^{-1} (W^{1-\eta}L)^{-\frac{1}{\eta}}. \quad (38)$$

The LHS of (38) is proportional to the marginal utility of leisure, and increases in L . The RHS is proportional to the product of the wage W with the marginal utility of consumption. As L increases, the wage declines. Wage income also declines because tasks are complements, making labor demand inelastic; because workers are hand-to-mouth, this translates to an increase in the marginal utility of consumption. When combined complementarities are strong ($\sigma + \eta < 1$), the increase in marginal utility of consumption dominates for L sufficiently large, making the RHS a U-shaped function of L , such that multiple solutions may exist. In contrast, if complementarities are weak then the dominant factor in determining the slope of the RHS is the decline in wages, and the solution is unique.

More formally, multiple steady states are possible if complementarities are strong because, as Propositions 2 and 4 establish, distinct asymptotic paths for K_o and K_w are consistent with the transversality conditions (3). In one possibility (Proposition 2) $\lim g_{K_i} = \eta(A_K - \delta_i - \rho)$ for $i = o, w$, with workers' transversality condition satisfied because labor income becomes a negligible part of their income; while in the other possibility (Proposition 4) $\lim g_{K_o} = \frac{\eta\theta}{1-\sigma}$, where workers don't hold capital ($K_{w,t} = 0$ for all t large enough) and their transversality condition is trivially satisfied.

3.2 The distributional consequences of automation

The stratification of households into “worker” and “capitalist” roles (Proposition 1, stemming from heterogeneity in δ_i) allows us to address the distributional consequences of automation. If automation (θ) is sufficiently slow, and task complementarities (σ) are sufficiently strong then workers'

share of the economy remains stable asymptotically, whether measured either by income or consumption, and even as the fraction of non-automated tasks approaches 0. Moreover, local increases in the speed of automation benefit workers by increasing their consumption growth, and by the same amount as GDP growth. At the same time, increases in the speed of automation shrink workers' share of the economy, as can be seen from Proposition 4.

Conversely, automation speeds above a threshold threaten workers more profoundly. In the case of weak complementarities, workers' consumption growth (weakly) decreases in the automation rate. Income and consumption inequality both explode, with workers' share of the economy asymptoting to zero. More positively: workers' consumption growth nonetheless remains positive, and as time passes they work vanishingly little, so even in this case workers' absolute living standards improve as more and more tasks are automated.

For strong complementarities, multiple steady states exist—one with a stable labor share, and one with capital dominance. Workers' consumption growth is strictly higher in the former, even though both GDP and capitalists' consumption grows strictly faster in the latter:

Lemma 5 *If stable labor share and capital-dominant steady states coexist then workers' consumption grows strictly faster, their labor shrinks to zero strictly faster, and capitalists' consumption grows strictly more slowly,¹² in the stable labor share steady state than in the capital-dominant one.*

As outlined above, multiplicity arises from an interaction between heterogeneity in δ_i and endogenous labor choice under strong complementarities. But even if complementarities are weak, the distributional consequences of automation depend on labor choice. Consumption-leisure complementarity implies that workers work less as automation progresses and the economy grows. Regardless of whether the labor share asymptotically vanishes, equilibrium hours worked do. In this, our analysis is consistent with naïve predictions that no jobs will remain for human workers, and that neglect the countervailing effect of the increasing value of labor in non-automated tasks.

Although one might be tempted to conclude that the asymptotic vanishing of hours worked makes it more likely that the labor share shrinks to zero, the multiplicity discussion above indicates that the reverse is in fact true. Complementarity across tasks ensures that labor income is *decreasing*

¹²Capitalists' consumption grows at the same rate in the two equilibria if (35) holds with equality.

in labor supply, and consumption-leisure complementarity ensures that endogenous labor choice pushes against capital dominance as the economy grows. To see this, consider briefly a perturbed version of our model, in which we exogenously specify workers' labor supply, instead of allowing it to be chosen optimally. Specifically, we suppose that the growth rate of workers' labor supply is exogenously set to $g_{L_w} \leq 0$, while capitalists' labor supply is exogenously set to zero, $L_{o,t} \equiv 0$. The case of constant labor supply by workers is simply the special case $g_{L_w} = 0$.

Proposition 5 *If labor choices are exogenous, the economy has a stable labor share if and only if*

$$\theta < (1 - \sigma) (\eta (A_K - \delta_o - \rho) - g_{L_w}). \quad (39)$$

Proposition 5 establishes that the faster workers' labor supply shrinks the *more* likely it is that the economy asymptotes to a stable labor share. That is: the equilibrium shrinkage of hours worked in our main analysis pushes against capital dominance, as opposed to engendering it. Economically: Just as capital dominance is hindered by faster capital accumulation (see above), it is promoted by faster labor growth (i.e., labor that shrinks towards 0 more slowly).

These observations prompt consideration of a policy to head off capital dominance, namely regulation that caps the number of hours worked, with the cap shrinking over time. Such a regulation would resemble France's 35-hour work-week, with the added stipulation that the 35-hour limit gradually fall. Section 4 discusses this and two other realistic policy interventions.

3.3 Heterogeneity in households' ability to hold and manage capital

To understand the effects of heterogeneity in households' ability to hold and manage capital, it is useful to consider the comparative static of equilibrium outcomes with respect to δ_w , corresponding to worker-households becoming worse at capital-management.

If automation is sufficiently fast ($\theta \gg (1 - \sigma) (A_K - \delta_w - \rho) + \eta (\delta_w - \delta_o)$), increases in δ_w reduce the consumption growth-rate of worker-households. The effect here is direct: the economy exhibits capital dominance, so even workers' income stems overwhelmingly from capital income, and a deterioration in workers' ability to hold and manage capital reduces consumption growth. Perhaps

less obvious: an increase in δ_w *also* leads hours worked to fall more slowly, which pushes the growth rate of labor income still lower.

If automation is slower, and complementarities are weak ($\sigma + \eta > 1$), increases in δ_w expand the range of parameters for which workers give up holding capital even when the economy exhibits capital dominance. This is the extensive margin manifestation of the direct effect just discussed.

The most striking effects of an increase in δ_w emerge if complementarities are strong. In this case, higher values of δ_w expand the set of parameters for which capital-dominant and stable-labor share steady states coexist. So in particular, higher values of δ_w expand the circumstances in which workers are subject to “rat-race” type dynamics, in which (relatively) they work hard because their consumption growth is slow, but consumption growth is slow because of ample labor supply.

4 Policy

As we have highlighted, heterogeneity in the ability (δ_i) to hold and manage capital means that high rates of automation engender high levels of inequality. In this section, we use our framework to consider two roles for policy. First, we ask: what is the best way for a government that cares about inequality to ensure that workers’ consumption share converges to some (arbitrary) share \mathcal{X} ? Second, we ask: in light of the different steady state outcomes that arise from heterogeneity in δ_i , can a suitably-designed policy allow a government to select its preferred steady state?

4.1 Targeting a consumption share \mathcal{X} for workers

Consider the case in which automation θ exceeds the threshold $(1 - \sigma)(A_K - \delta_o - \rho)$, so that, absent policy, workers’ consumption share of the economy converges to 0. Suppose that a government sets policy to target a strictly positive consumption share \mathcal{X} for workers; formally,

$$\mathcal{X} = \lim \frac{\lambda_w C_{w,t}}{Y_t}.$$

We compare the efficacy of three possibilities, the first two of which in particular are commonly discussed. First, a government can tax income and redistribute the proceeds. Second, a government

can take steps to directly impede the pace of automation. Third, a government can cap hours worked (France’s 35-hour week is the best-known example). We focus on the case in which capitalists are a small fraction of the population ($\lambda_o \approx 0$), which we take to be the relevant case.

4.1.1 Tax-and-redistribute

The government must raise tax revenue of $\mathcal{X}Y_t$. Because capitalists are a small fraction of the population, the government can then redistribute this revenue via a “basic income” policy of paying $\mathcal{X}Y_t$ to each household.

Asymptotically, all income takes the form of returns to capital held by capitalists, and so the government raises this revenue via a tax τ on capital, satisfying $\tau K_t = \mathcal{X}Y_t$. Under capital dominance, $\lim \frac{A_K K_t}{Y_t} = 1$, and so $\tau = A_K \mathcal{X}$. (Equivalently, the government could tax the return to capital at rate \mathcal{X} .)

Taxation of capital is equivalent to a level-increase in δ_o and δ_w . This reduces incentives to accumulate capital, and accordingly lowers the growth of both capital and national income. The effective increase in δ_o and δ_w also pushes the economy towards capital dominance (see the threshold condition (35)). The reason is that capital-taxation reduces the growth rate of capital, making it scarcer, and raising its equilibrium return. As discussed following Proposition 3, the net effect is to promote capital dominance (given task-complementarity $\sigma < 1$).

From Propositions 2 and 3, the end result is that, given a target \mathcal{X} , the post-redistribution consumption of workers grows at rate

$$\eta (A_K (1 - \mathcal{X}) - \delta_o - \rho). \tag{40}$$

4.1.2 Automation retardation

Alternatively, a government can take steps to directly reduce the pace of automation (pejoratively, “Luddite” policies). Specifically, if the government lowers the automation rate θ —or, equivalently, the rate of automation adoption—to below $(1 - \sigma)(A_K - \delta_o - \rho)$ then an equilibrium with a stable labor share arises. Because capitalists are a small fraction of the population, the labor share coincides with the ratio of workers’ consumption to national income (recall that workers don’t hold capital,

while capitalists may or may not work). The more the automation rate is slowed, the greater the labor share is, though at the expense of lower growth rates. Specifically, from (17), (36) and (37), the stable labor share is related to the consumption growth rate and the return on capital via

$$\lim X = 1 - \left(\frac{\delta_o + \rho + \frac{g_{C_w}}{\eta}}{A_K} \right)^{1-\sigma} = 1 - \left(\frac{\bar{F}_K}{A_K} \right)^{1-\sigma}. \quad (41)$$

Hence the consumption growth rate associated with a target worker share of \mathcal{X} is

$$\lim g_{C_w} = \eta \left(A_K (1 - \mathcal{X})^{\frac{1}{1-\sigma}} - \delta_o - \rho \right) = \eta (\bar{F}_K - \delta_o - \rho). \quad (42)$$

4.1.3 Caps on hours worked

A third alternative is for a government to impose a (dynamically-moving) cap on hours worked. France’s 35-hour week is the best known example; and justifications for such policies along the lines of “sharing a limited supply of work” or “protecting workers from capitalists” are consistent with the economic forces in our analysis. From Proposition 5, a labor regulation that implements a sufficiently negative g_{L_w} would shift the economy to one with a stable labor share. The proof of Proposition 5 establishes that, by suitable choice of g_{L_w} , a government can attain any asymptotic level of capital return $\bar{F}_K < A_K$ that it desires. The capital return \bar{F}_K in turn determines the growth rate of the economy via the capitalists’ intertemporal condition $g_Y = g_{K_o} = \eta (\bar{F}_K - \delta_o - \rho)$ and the asymptotic labor share via (41).

4.1.4 Comparison

All three of the policies just discussed introduce distortions that lower the growth rate of national income. Nonetheless, the comparison between them is straightforward: Automation-retardation and caps-on-hours are equivalent, in that the two give exactly the combination of worker-consumption shares and growth rates; and both policies are strictly dominated by taxation-and-redistribution.

A rough intuition is as follows. From (17), reducing the asymptotic capital share below 1 via automation-retardation entails reducing the asymptotic return to capital \bar{F}_K below its maximal value of A_K . But this is an expensive way to reduce the capital share, as follows. On the one hand,

each percentage point reduction in \bar{F}_K reduces asymptotic consumption growth (of both capitalists and workers) by η percentage points; see the intertemporal optimality condition (22). But on the other hand, a reduction in \bar{F}_K is naturally associated with an *increase* in the capital-output ratio $\frac{K}{Y}$. Because the asymptotic capital share is, by definition, $\bar{F}_K \lim \frac{K}{Y}$, this means that there is incomplete pass-through of a reduction in \bar{F}_K on the capital share. In contrast, a percentage point of capital taxation has same effect of reducing consumption growth by η percent, but enjoys complete pass-through in terms of its effect on capitalists' share of the economy.

Like the rest of the paper, this analysis assumes that the fraction of unautomated tasks for which automation becomes feasible (θ) is exogenous. Nonetheless, the government can slow down the decline in the share of labor-performed tasks through employment-favoring policies (e.g., requiring that certain tasks are performed by humans). Such policies drive a wedge between the fraction of tasks that *can* be automated at any point in time (α_t) and those that *are* ($\tilde{\alpha}_t$).¹³

If the innovation that drives automation is endogenous and embedded in capital, then capital-taxation may additionally reduce θ . Our analysis has the benefit of cleanly separating this effect from those arising from endogenous factor returns. In this case, however, the tax-and-redistribute policy is bundled with automation retardation in practice. We note two things in this context. First, this channel only affects the above discussion if capital taxation lowers θ below the capital-dominance threshold. Otherwise, $F_K = A_K$ and the above analysis goes through. Second, if capital taxation does reduce θ below the capital-dominance threshold, the above policy *comparison* remains unchanged. Heuristically, the tax-and-redistribute policy then becomes bundled with automation retardation and the required tax rate falls because some of the redistributive effect is achieved through the (more expensive) retardation channel. But the cost of this policy is then a weighted average of that of the two channels and, thus, smaller than the cost of pure automation retardation.

4.2 Policy for steady state selection

A second task for policy is to select between the multiple steady states that are engendered by heterogeneity in δ_i . Specifically, suppose that complementarities are strong, the pace of intermediation

¹³In practice, policies targeted at increasing employment have often taken the form of investment incentives. In the data, these achieve the opposite effect in that they *accelerate* the substitution of routine labor with capital (Tuzel and Zhang, 2021), mirroring an acceleration in automation or its adoption in our framework.

is intermediate, and heterogeneity sufficiently pronounced that

$$(A_K - \delta_w - \rho) + \frac{\eta}{1 - \sigma}(\delta_w - \delta_o) < \frac{\theta}{1 - \sigma} < (A_K - \delta_o - \rho), \quad (43)$$

so that stable-labor share and capital-dominant steady states coexist. From Lemma 5, workers have a higher consumption share in the former case, while overall growth is higher in the latter case.

What can a government do to select the steady state with the stable labor share, thereby prioritizing workers? Recall the tax-and-redistribute policy outlined above and suppose that the government guarantees a capital-tax funded top-up of wages to match the undistorted asymptotic wage growth in the stable labor share steady state, $\lim g_W = \frac{\theta}{1 - \sigma}$.

Given intratemporal optimality and the hand-to-mouth budget constraint, labor supply under this policy matches that of the same equilibrium labor supply matches that of the stable labor share equilibrium, as does the labor income and, hence, the labor share, χ . From (37), this implies

$$\chi = 1 - \left(\frac{\delta_o + \rho + \frac{\theta}{1 - \sigma}}{A_K} \right)^{1 - \sigma},$$

and the associated tax on capital required to fund this transfer is given by

$$\tau = \frac{Y_t}{K_t} \left(\chi - \frac{W_t L_t}{Y_t} \right). \quad (44)$$

Proposition 6 *Under policy (44), only the stable labor share steady state exists. The tax revenue collected asymptotes to zero.*

The economic force behind Proposition 6 is that, as discussed above, the capital-dominant steady state exists because of a self-fulfilling “rat race” in which low-consumption workers supply lots of labor, in turn driving income and hence consumption down. The tax-and-redistribute policy specified above prevents this rat race from occurring by directly boosting workers’ consumption.

If the government instead wishes to select the higher-growth-high-inequality capital-dominant steady state, then it needs to employ essentially the reverse strategy, namely reducing rather boosting workers’ consumption. This can be achieved via a deadweight tax on labor income; similar to Proposition 6, the incidence of this tax asymptotes to a negligible share of the economy:

Proposition 7 *There exists a deadweight tax on labor income such that the only steady state features capital dominance. The deadweight cost relative to output asymptotes to zero.*

5 Discussion

5.1 Automation, “robo-advising,” and the return wedge $\delta_w - \delta_o$

Automation raises aggregate growth rates. As emphasized, the concern with higher rates of automation is that the benefits flow disproportionately to capitalists. Recall that workers differ from capitalists only in their lower ability to hold capital, captured by the financial frictions wedge $\delta_w - \delta_o$. It is precisely this wedge that prevents workers from fully benefiting from faster rates of automation.

These observations raise the intriguing possibility that automation itself will provide the cure to the ills it generates, via a reduction in the financial frictions wedge $\delta_w - \delta_o$. As we have emphasized, “financial frictions” should be broadly interpreted. For example: if the wedge $\delta_w - \delta_o$ stems from differences in knowledge and expertise, low-cost and high-quality robo-advising would reduce and perhaps even eliminate the wedge. Alternatively, automation holds the potential to lower the cost of financial intermediation, leveling the playing field between low- and high- δ households. And if the wedge $\delta_w - \delta_o$ stems from differences in self-control, automation again holds the potential to allow high- δ households to delegate their financial decisions to a digital household “manager,” thereby ameliorating the consequences of self-control differences.

To address this possibility, we generalize our baseline model to make the worker-depreciation rate δ_w time-varying, reflecting automation of financial services along the lines discussed above,

$$\delta_{w,t} \equiv \delta_o + \Delta_t. \tag{45}$$

For transparency, we continue to assume the capitalist-depreciation rate δ_o is time-invariant.

The question we are interested in is whether the automation-induced dynamics of the financial frictions wedge Δ_t ameliorate the problems associated with capital dominance. Accordingly, for the rest of this section we assume that the economy’s parameters are such that if the wedge Δ_t remains

fixed at its current value then the economy asymptotes to capital dominance.¹⁴ From Propositions 2 and 3, workers avoid relative immiseration only if the wedge Δ_t asymptotically vanishes.

While various assumptions are possible for how automation will affect the wedge Δ_t , a natural one in the context of the rest of the paper is that the wedge will shrink exponentially towards zero, in common with the set of unautomated production tasks. Thus, suppose that for some $\chi > 0$

$$g_{\Delta,t} = \frac{\partial}{\partial t} \ln(\Delta_t) = -\chi. \quad (46)$$

From intertemporal optimality (22), at any time t at which workers hold capital, the difference in growth rates of marginal utility coincides with financial friction wedge,

$$\frac{\partial}{\partial t} \ln MU_{C_w,t} - \frac{\partial}{\partial t} \ln MU_{C_o,t} = \delta_{w,t} - \delta_o = \Delta_t. \quad (47)$$

Recall from (27) that because leisure approaches its upper bound, this difference in marginal-utility growth rates can be expressed directly as a difference in consumption growth rates,

$$g_{C_o,t} - g_{C_w,t} \rightarrow \Delta_t. \quad (48)$$

Let date 0 be an arbitrary reference date. A worker's consumption as a fraction of a capitalist's consumption, $\frac{C_{w,t}}{C_{o,t}}$, equals

$$\begin{aligned} \frac{C_{w,0}}{C_{o,0}} \exp\left(\int_0^t (g_{C_w,s} - g_{C_o,s}) ds\right) &\approx \frac{C_{w,0}}{C_{o,0}} \exp\left(-\int_0^t \Delta_s ds\right) \\ &= \frac{C_{w,0}}{C_{o,0}} \exp\left(-\Delta_0 \int_0^t \exp(-\chi s) ds\right). \end{aligned} \quad (49)$$

Importantly, the final expression is bounded away from 0 even asymptotically, implying that, if financial frictions shrink towards zero exponentially, workers escape relative immiseration.

Although (49) implies that workers avoid immiseration, it is still the case that capital dominance

¹⁴For the case of strong complementarities ($\sigma + \eta < 1$) we assume, moreover, that capital dominance is the unique equilibrium outcome under current parameter values. Section 3.3 already analyzes the case in which capital dominance and a stable labor share coexist as equilibrium outcomes, and establishes that even modest declines in the wedge Δ_t potentially eliminate the capital dominance equilibrium.

potentially leads to a high degree of worker-capitalist inequality. To see this, note that evaluation of (49) as $t \rightarrow \infty$ delivers

$$\lim \frac{C_w}{C_o} = \frac{C_{w,0}}{C_{o,0}} \exp\left(-\frac{\Delta_0}{\chi}\right). \quad (50)$$

A reasonable starting point to quantify (50) is to assume that technological progress eliminates financial frictions at the same speed as it automates labor-produced tasks, viz., $\chi = \theta$. Looking ahead, we argue in Section 6 that capital dominance requires a speed of automation θ of approximately 2.4% (assuming $\sigma \approx 0.5$) or more; and moreover, that this speed is considerably above estimates of the current rate. We also argue that the current level of Δ_t is on the order of 2%. Consequently, if the speed of automation rises enough to potentially trigger capital dominance, (50) suggests that the technology-driven asymptotic elimination of financial frictions would stabilize worker consumption as a fraction of capitalist consumption at approximately $\exp\left(-\frac{2}{2.4}\right) \approx 43\%$ of its current level.

In summary: there is some hope that automation will cure its own ills by exponentially shrinking the frictions underlying the return difference between capitalists and workers. However, even if complete (relative) immiseration is avoided, capital dominance still implies a considerable rise in worker-capitalist consumption inequality.

As a final point: The asymptotic disappearance of the financial frictions wedge Δ_t is necessary but not sufficient for workers to escape immiseration. From (49), the avoidance of worker immiseration requires

$$\int_0^t \Delta_s ds < \infty. \quad (51)$$

Condition (51) is satisfied if the wedge Δ_t approaches 0 at exponential speed. Conversely, however, condition (51) fails if the wedge Δ_t approaches 0 at rate $1/t$; and in this case, workers again end up with a vanishingly small share of the economy. As a specific instance of a force that may slow the speed at which automation reduces the wedge Δ_t , Greig et al. (2024) empirically study robo-advising, and identify frictions to its adoption.

5.2 OLG-heterogeneity and endogenous automation

In light of our focus on the long-run limit as $\alpha_t \rightarrow 1$, two of our assumptions deserve further discussion: the persistent capital-holding abilities of the (infinitely-lived) agents in our model and

the exogenous speed of automation. We discuss these assumptions in turn.

We assume that our heterogeneous agents are infinitely-lived and keep their exogenous types (δ_o or δ_w) forever. This set-up is isomorphic to an overlapping-generations model with stochastic death, perfect bequest motives, and perfect persistence in types. Relaxing the bequest motive is equivalent to raising the time-preference rate ρ in our set-up. A higher rate of time preference reflects stochastic mortality and would lead to lower individual savings and slower aggregate capital accumulation. Thus, an OLG-setting with imperfect bequest motive makes capital dominance more likely than in our set-up.

Relaxing persistence in types implies that in every generation, some worker-type agents have inherited wealth from their capitalist parents. In this case, the wealth-weighted average rate of depreciation rate that enters the capital-dominance condition rises to $\tilde{\delta} > \delta_o$. Thus, weakening persistence in types, either by letting δ_i follow a Markov process for an infinitely-lived agent, or by modeling agents as overlapping generations with imperfectly heritable types, also makes capital dominance more likely than in our set-up.

Our assumption that the rate of automation, θ , is exogenous reflects a benchmark case marked by an inexorable “march of progress,” and allows us to focus on the long-run consequences of automation. How would our conclusions change if instead the rate of automation were endogenous? A key consideration is whether innovation in automating further tasks is capital or labor-intensive. If innovation is labor-intensive, then the rate of automation will tend to rise in the ratio of marginal products of capital and labor, $\frac{F_{K,t}}{F_{L,t}}$. In this case, endogenous automation would amplify exogenous variation in automation rates. Specifically, high exogenous rates of automation are associated with capital dominance and high ratios $\frac{F_{K,t}}{F_{L,t}}$, thereby endogenously further increasing the automation rate. Conversely, low rates of automation are associated with a stable labor share and low ratios $\frac{F_{K,t}}{F_{L,t}}$, thereby endogenously further decreasing the automation rate.

However, if instead innovation is capital-intensive, then parallel arguments suggest that endogenizing automation dampens exogenous variation in automation rates.

5.3 r , g , and capitalist-worker inequality

Ceteris paribus, higher rates of return on capital favor capitalists at the expense of workers, a point emphasized by Piketty (2017). Here, we briefly discuss our analysis's implications for the asymptotic relation between the net return on capital, which we label $r = R - \delta_o$,¹⁵ the growth rate of the economy, g_Y ; and capitalist-worker consumption inequality induced by heterogeneity in δ_i . From (17) and our equilibrium characterization: in a stable labor share equilibrium

$$r = \rho + \frac{\theta}{1 - \sigma} \text{ and } \frac{r}{g_Y} = \frac{\rho + \frac{\theta}{1 - \sigma}}{\frac{\eta\theta}{1 - \sigma}}$$

while in a capital-dominant equilibrium

$$r = A_K - \delta_o \text{ and } \frac{r}{g_Y} = \frac{A_K - \delta_o}{\eta(A_K - \delta_o - \rho)}.$$

The asymptotic interest rate is increasing in the speed of automation, since faster automation keeps capital scarcer. Moreover, by Corollary 2, the interest rate in a capital-dominant equilibrium would exceed today's rate, since $A_K \geq F_{K,t}$.

Thus, on the one hand, faster rates of automation are associated both with higher values of r and with greater capitalist-worker consumption inequality, consistent with the partial equilibrium reasoning that higher rates of return on capital favor capitalists.

On the other hand, faster rates of automation are associated with lower ratios of r to g_Y . The reason is simply that faster automation increases output growth proportionately more than it increases the return to capital. Combined with the fact that r exceeds g_Y (as it must in any setting in which capital and output grow at the same rate, and the transversality condition (3) holds), it follows that while both r and g_Y increase in the rate of automation θ , the ratio r/g_Y decreases.

Consequently, the ratio r/g_Y is negatively related to capitalist-worker inequality, at least asymptotically. This is true both as one varies the automation rate θ , and also as one moves across the different steady states that coexist in the case of strong complementarities ($\sigma + \eta < 1$).

¹⁵We evaluate r using the capitalists' $\delta_i = \delta_o$ as capitalists asymptotically hold all capital in all equilibria.

5.4 Observable trends

While we predominantly examine asymptotic factor shares in the full-automation limit, the key forces in our analysis also speak to three long-run empirical trends: (i) the decline in hours worked, (ii) the recent rise in the capital share, which our model naturally connects to a decline in TFP growth, as well as its previous stability, and (iii) a reallocation in output shares towards services.

Time spent working converges to zero in all equilibria of our model, both for workers and capitalists. As such, our analysis predicts a long-term decline in hours worked, even away from the limit, consistent with empirical observation (Boppart and Krusell, 2020, and references therein). The prediction stems in part from the preference-specification with consumption and leisure as complements, $\eta < 1$, which does not belong to the class proposed by King et al. (1988) to generate stable hours worked despite consumption growth. Another driving force, which interacts with consumption-leisure complementarity, is task complementarity ($\sigma < 1$). As the economy accumulates capital but the per-capita time endowment stays fixed, labor-produced tasks become scarce and wages rise. That is, with task complementarity, automation is labor-augmenting (see also AJJ) and generates wage growth. With consumption-leisure complementarity, wage growth translates partially to leisure growth.

Regarding a rising capital share and falling TFP-growth, Lemma 1 implies that TFP growth in our economy is given by

$$g_{TFP} = \frac{\theta}{1 - \sigma} \left(1 - \frac{1 - X_t}{\alpha_t} \right). \quad (52)$$

Philippon (2023) argues that, empirically, TFP has grown linearly, which implies that g_{TFP} has dropped. The last term in (52) implies that our model features declining TFP growth if the capital share grows faster than the share of automated tasks, α_t . Observe first that the share of automated tasks grows at a rate $g_{\alpha,t} = \theta(1 - \alpha_t)/\alpha_t$. It follows from equation (12) that (given $\sigma < 1$) the capital share grows faster than the share of automated tasks, and hence the growth rate of TFP declines, if and only if the marginal product of capital rises. US growth rates of capital, output, and the capital share from 1970-2019 satisfy this condition (see Table 1).

Regarding the stable factor shares observed in the immediate post-WWII period, we note that the labor share in our model depends crucially on the relative growth rates of capital and labor.

This growth differential was around 1 percentage point larger in the 1950-1970 period than between 1970 and 2019. Our model is thus consistent with a period of labor-share stability prior to its eventual decline (see Appendix E).

Lastly, our analysis has implications for the fraction of GDP stemming from each non-automated task or “sector” prior to its automation, namely, $X_t/(1 - \alpha_t)$. Hence, the growth rate of each non-automated task’s GDP share prior to automation is

$$g_{X,t} + \theta. \tag{53}$$

It follows that non-automated sectors grow faster than the overall economy whenever θ outweighs the rate of decline in the labor share at any point in time. (Note that, at least asymptotically, this condition is weaker than the condition for a stable labor share. That is: Even with capital dominance, the growth rate of as-yet non-automated sectors may exceed that of the overall economy.)

In the preliminary calibration we present in the following section, expression (53) is indeed positive for the US in the 1970-2019 period. It is natural to think of non-automated tasks largely as services, which Boppart (2014) shows have seen steadily rising expenditure shares in the US. Among the industries that have outgrown the overall economy at the fastest pace in recent decades, many naturally come to mind as examples of non-automated tasks, such as education, healthcare, restaurants, or performing arts. The industry with the biggest relative *decline* is manufacturing.¹⁶

6 A preliminary calibration

We make a first pass at calibrating our analysis, and in particular, assessing whether the economy is in the stable labor share or the capital dominance region. By its nature, this exercise is highly speculative. But with that caveat, our calibration suggests that the current speed of automation is too slow to generate capital dominance.

¹⁶Using BEA data by industry *Manufacturing* lags total cumulative growth in value added between 1998 and 2021 by 32%, while *Food services* (25%), *Performing arts, spectator sports, and related activities* (25%), *Health care* (26%), and *Educational services* (31%) have all grown faster than total value added. Hubmer (2023) confirms quantitatively that these sectors—unlike manufacturing—have above-average labor shares.

Recall that capital dominance arises as a unique equilibrium if and only if:

$$\frac{\theta}{1-\sigma} > A_K - \delta_o - \rho. \quad (54)$$

In addition, a capital-dominant steady state coexists with a stable labor share one if complementarities are strong ($\sigma + \eta < 1$) and

$$\frac{\theta}{1-\sigma} + \frac{1-\sigma-\eta}{1-\sigma} (\delta_w - \delta_o) > A_K - \delta_o - \rho. \quad (55)$$

6.1 Capital dominance as a unique steady state?

We first assess whether capital dominance arises as a unique steady state, i.e., whether (54) holds.

Evaluation of the RHS of (54) requires an estimate of A_K . Our main approach is to use the fact that A_K is bounded below by the marginal product of capital (see Corollary 2), which can in turn be estimated from observables:

$$A_K \geq F_{K,t} = \frac{1-X_t}{\frac{K_t}{Y_t}}. \quad (56)$$

Turning to the LHS, we first note that substitution of (11) into (13) and straightforward manipulation links the automation rate θ to observable growth rates of output, labor, and labor share, and to the elasticity of substitution across tasks (σ):

$$\theta = (1-\sigma)(g_{Y,t} - g_{L,t}) - \sigma g_{X,t}. \quad (57)$$

As noted, the elasticity σ coincides with the production-based elasticity of substitution between capital and labor, and a significant literature is devoted to its estimation (e.g., Chirinko, 2008; Oberfield and Raval, 2021). Viewing the output aggregation into a single good through a consumption lens, the relevant elasticity of substitution is one across consumption goods (e.g., Nordhaus, 2021). Consequently, (57) links θ to observables and existing estimates of the elasticity σ .

Remark: Although we have abstracted away from growth in labor productivity A_L in our main analysis, a nice feature of our calibration approach is that it is invariant to allowing for such growth. That is: in both the threshold condition (54) and the estimation equation (57), the automation rate

θ is replaced by an “augmented” automation rate $\tilde{\theta} \equiv \theta + (1 - \sigma) g_{A_L}$. See Online Appendix G.

For inputs, we use the following from National Income Accounts (as of 2019) and the US Census Bureau, all scaled by the working population (see Appendix B for details):

$$\left(g_Y, g_L, g_K, g_X, g_{1-X}, X, \frac{K}{Y}, \delta_o, \rho \right) = (1.36\%, -0.19\%, 0.99\%, -0.17\%, 0.28\%, 59.7\%, 3.63, 4.32\%, 2\%)$$

Figure 2 displays the rate of automation θ (calculated from (57)) and the key ratio $\frac{\theta}{1-\sigma}$ as a function of σ . Based on this speculative exercise, our analysis implies that the economy will not asymptote to a unique capital-dominant equilibrium, and that, instead a stable labor share equilibrium will exist, as follows. First, the lower bound (56) for A_K implies

$$A_K - \delta_o - \rho \geq \frac{40.3\%}{3.63} - 4.32\% - 2\% = 4.79\%. \quad (58)$$

From Figure 2, the ratio $\frac{\theta}{1-\sigma}$ only approaches this bound if the elasticity parameter σ is close to 1 (it exceeds the bound for $\sigma \geq 0.95$), that is, outside the typical range of empirical estimates.¹⁷

An alternative and independent approach to calibrating the automation rate builds on Acemoglu (2024)’s estimate that approximately 5% of GDP will be disrupted by AI in the next 10 years, and yields $\theta \approx 0.9\%$ (details in Appendix B). This both matches the value estimated using (57) and an input of $\sigma \approx 0.5$; and falls below the threshold for capital dominance provided that the elasticity of substitution across tasks, σ , is below 0.81, which encompasses typical empirical estimates.

Finally: Appendix B explores two further alternative calibrations, both of which lead to the same conclusion that the speed of automation falls short of the capital-dominance threshold.

6.2 Capital dominance as one of multiple steady states?

Next, we assess whether capital dominance arises as one of two coexisting steady states, i.e., evaluate whether the combination of $\sigma + \eta < 1$ and (55) holds.

¹⁷In his synthesis of the literature on capital-labor substitution in production, Chirinko (2008) writes that “the weight of the evidence suggests a value of σ in the range of 0.40–0.60.” In a recent estimation for the manufacturing sector, Oberfield and Raval (2021) place it at 0.5–0.7. Given the dual role of σ in capturing both technology- and preference-based complementarities, we also note that estimates based on consumption expenditure similarly point towards gross complementarity (Nordhaus, 2021).

Relative to (54), condition (55) features the wedge in the ability to hold and manage capital between workers and capitalists, $\delta_w - \delta_o$, and the consumption-leisure substitutability parameter η .

Fagereng et al. (2020) estimate a return differential of around 1.5 percentage points within safe assets and around half a percentage point within risky assets. In US data, Smith et al. (2023) estimate a gap of close to three percentage points within fixed income assets between the bottom 99.9% and the top 0.01% of the wealth distribution. Through the lens of intermediation rents, Philippon (2015) estimates an intermediation cost of 1.5–2% of intermediated assets. Below, we consider input values for the return wedge $\delta_w - \delta_o$ of 1.5% to 4.5%.

For η , the elasticity of substitution between consumption and leisure, we use the following result that expresses this elasticity in terms of the Frisch elasticity of labor supply and the IES:

Lemma 6 *The elasticity of substitution between consumption and leisure (η) is related to the Frisch elasticity of labor supply ($\frac{\Delta \ln L}{\Delta \ln W}$) and the IES ($\approx \frac{1}{\gamma}$) according to*

$$\eta = \left(\frac{WL}{C} + \frac{L}{1-L} \right) \frac{\Delta \ln L}{\Delta \ln W} - \frac{1}{\gamma} \frac{1-L}{L} \frac{WL}{C}. \quad (59)$$

For the Frisch elasticity, we use the value 0.32 reported in Chetty et al. (2013), obtained from averaging estimates of previous studies.¹⁸ The income-to-consumption ratio $\frac{WL}{C}$ is approximately 1.1. Hence even if one takes a conservative view of what constitutes leisure, leading to an input $\frac{L}{1-L} \approx 2$, and even if one believes the IES is very small, Lemma 6 still implies $\eta < 1$. Adopting a more expansive view of leisure ($\frac{L}{1-L}$ lower) and/or a higher IES reduces the estimate of η .

With respect to η , we conclude that micro-estimates indeed suggest that $\eta < 1$, as assumed throughout. Moreover, values of η below 0.5 are consistent with reasonable inputs in Lemma 6, and such values imply that the strong complementarity condition $\sigma + \eta < 1$ is likely satisfied.

We are now ready to consider condition (55), determining whether capital dominance coexists with a stable labor share as a possible steady state. First, consider the case in which the return wedge is at the lower end of estimated values, $\delta_w - \delta_o = 1.5\%$. Note that the wedge term in (55) is bounded above by $\delta_w - \delta_o$. From Figure 2, it follows that (55) holds only for values of task-complementarity σ that are very close to 1 (approximately 0.9 or higher), which is well outside the

¹⁸See Table 1 in (Chetty et al., 2013); the range of estimates is 0.18 – 0.43.

range of empirical estimates (see footnote 17).

Second, consider the case in which the return wedge is at the upper end of estimated values, $\delta_o - \delta_w = 3\%$. The top panel of Figure 3 plots the LHS of (55) for $\eta = 0.025, 0.05, 0.1$, with the domain of σ restricted to ensure the strong complementarity condition $\sigma + \eta < 1$ holds. The figure demonstrates that (55) holds for task complementarity $\sigma < 0.9$ *only* if consumption-leisure complementarity is extremely strong, $\eta < 0.05$, i.e., close to Leontief.

Third, consider the case of a return wedge that exceeds estimated values, $\delta_w - \delta_o = 4.5\%$. The bottom panel of Figure 3 is analogous to the top panel, and plots the LHS of (55) for $\eta = 0.05, 0.1, 0.25$. In this case, the return wedge is large enough that the LHS of (55) is decreasing rather than increasing in the task-complementarity parameter σ . For very strong consumption-leisure complementarity ($\eta < 0.05$), the capital dominance condition (55) holds. Even for less extreme (though still strong) complementarity inputs, such as η in the range of $0.1 - 0.15$, the capital dominance condition (55) holds for a wide range of task-complementarity inputs that is consistent with empirical estimates. However, consumption-leisure complementarity weaker than $\eta = 0.28$ is inconsistent with the capital dominance condition (55), *regardless* of the strength of task-complementarity σ .

To conclude: The “multiplicity” capital dominance condition (55) is naturally easier to satisfy than the “unique” capital dominance (54). Nonetheless, (55) holds only if at least one of the inputs σ and $\delta_w - \delta_o$ is chosen aggressively, relative to available estimates; and even in these cases, (55) holds only if consumption-leisure complementarity is very strong. As such, the calibration suggests that the recent pace of automation is likely too slow to deliver capital dominance.

6.3 Lower bounds for the asymptotic labor share

Finally, and more briefly, our calibration exercise also delivers a lower bound for the asymptotic labor share. To obtain this lower bound, we apply Proposition 4, using as an input for A_K the lower bound calculated from the observed return to capital $F_{K,t}$. As task-complementarity strengthens from $\sigma = 0.7$ to $\sigma = 0.4$ (focusing here on the range most consistent with empirical estimates, see footnote 17), the estimated lower bound for the asymptotic labor share increases from approximately 8.5% to approximately 18%. Especially in light of the low values of these estimates, it is worth

emphasizing that these are lower bounds, not point estimates. Tighter bounds would require better estimates of A_K than we have been able to construct.¹⁹

7 Conclusion

Automation shifts income from labor to capital. Recent progress in artificial intelligence raises the prospect that, asymptotically, all tasks will be automated. In such an economy the ability to hold and manage capital is a key determinant of individual outcomes. We take seriously the extensive empirical evidence that households differ in their ability to generate capital income. We study how this heterogeneity not only shapes consumption inequality in the full-automation limit, but also affects the labor supply and savings choices that shape equilibrium wages and capital accumulation.

We show that simple heterogeneity in investment returns endogenously stratifies agents into capitalists and workers. Their labor and savings choices, in turn, determine the extent to which asymptotically full automation shifts factor shares in the first place. With complementary tasks, the labor share approaches a strictly positive limit if automation is sufficiently slow relative to capital accumulation. In this case, even vanishingly few unautomated tasks generate sufficient labor income to prevent relative worker immiseration. For intermediate automation rates, a high-inequality, capital-dominant steady state coexists with a lower-inequality steady state in which labor retains a strictly positive income share. In this region, workers' inferior investment returns lead them to work more, thus promoting capital dominance and workers' reliance on capital income.

Our model provides a natural setting for policy analysis, and implies that the negative consequences of capital dominance are better ameliorated via taxation-funded "basic income" than by the deliberate retardation of automation. In the multiplicity region, a simple policy can select the low-inequality steady state with an equilibrium tax share of zero.

¹⁹The parameter A_K is related to the current labor share X_t and return to capital $F_{K,t}$ by $A_K = \left(\frac{\alpha_t}{1-X_t}\right)^{\frac{1}{1-\sigma}} F_{K,t}$. But this formula requires a credible estimate of the fraction of automated tasks α_t .

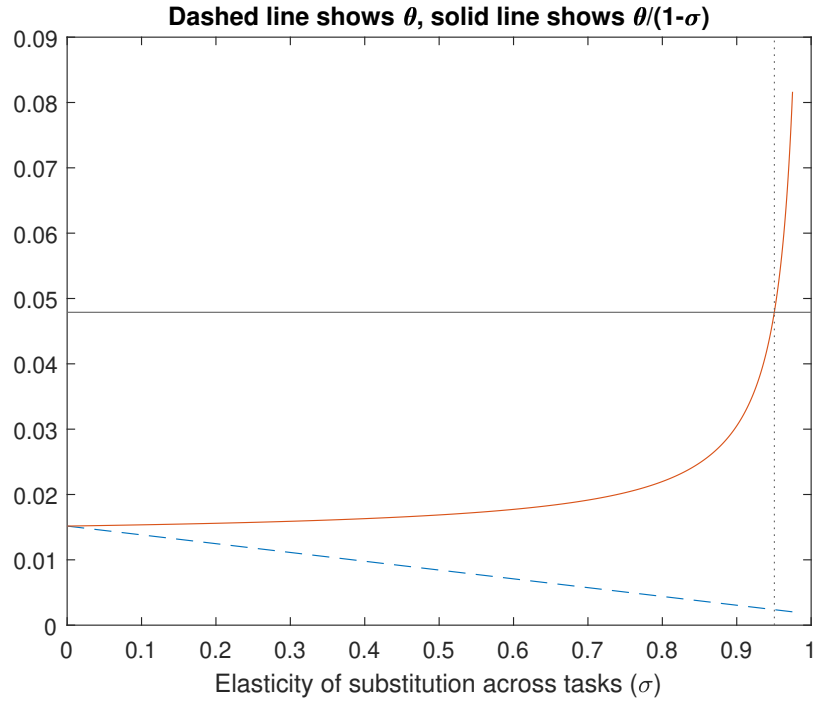
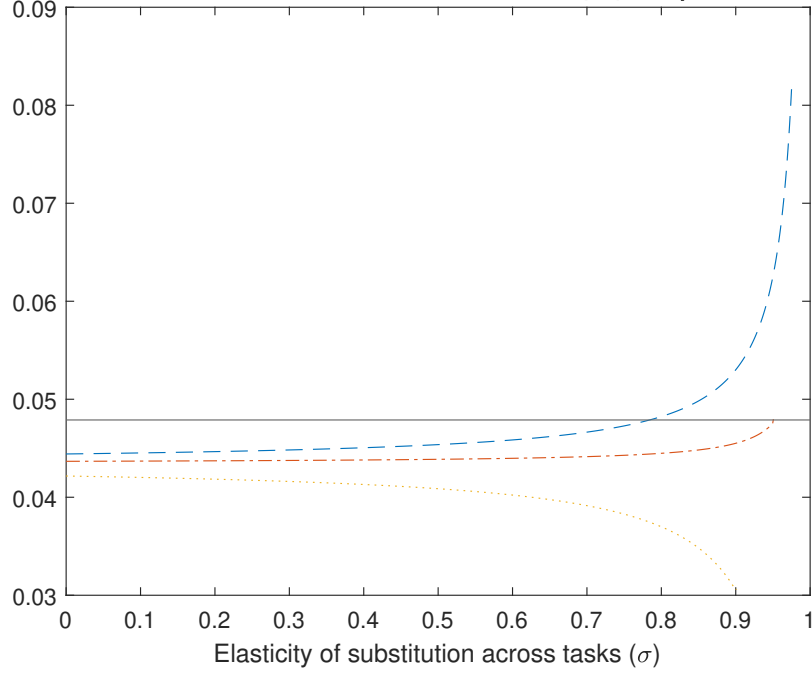


Figure 2: θ (dotted, blue) and $\frac{\theta}{1-\sigma}$ (solid, orange) as functions of the elasticity of substitution between tasks, σ . The automation rate θ is inferred from (57). The gray, horizontal line marks the lower bound on $A_K - \delta_o - \rho$ from (58).

Dashed, dash-dot and dotted lines show LHS of (44) for $\eta = 0.025, 0.05, 0.1$



Dashed, dash-dot and dotted lines show LHS of (44) for $\eta = 0.05, 0.1, 0.25$

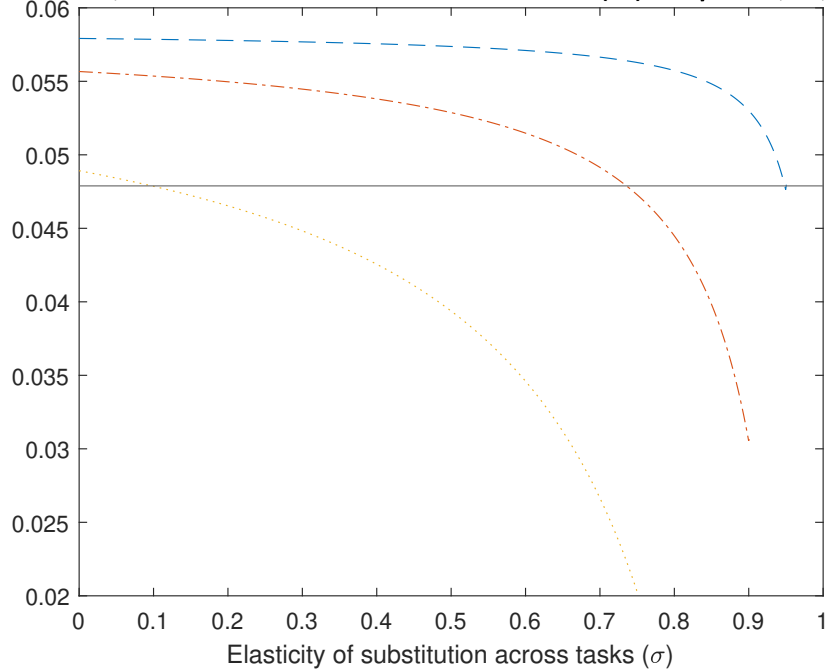


Figure 3: LHS of (55) as functions of the elasticity of substitution between tasks, σ for different values of η ; top panel: 0.025 (blue, dashed), 0.05 (orange, dash-dot), 0.1 (yellow, dotted), bottom panel: 0.05 (blue, dashed), 0.1 (orange, dash-dot), 0.25 (yellow, dotted). The gray, horizontal line marks the lower bound on $A_K - \delta_o - \rho$ from (58).

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A Proofs

For use in results after Lemma 1: note that Lemma 1, (10) and (11) imply

$$g_{R,t} = \frac{1}{\sigma} \left(X_t (g_{L,t} - g_{K,t}) + \frac{\theta}{1-\sigma} \left(1 - \frac{1-X_t}{\alpha_t} \right) + \theta \frac{1-\alpha_t}{\alpha_t} \right) \quad (\text{A-1})$$

$$g_{W,t} = \frac{1}{\sigma} \left((1-X_t) (g_{K,t} - g_{L,t}) + \frac{\theta}{1-\sigma} \left(\sigma - \frac{1-X_t}{\alpha_t} \right) \right). \quad (\text{A-2})$$

Output growth (15), $\alpha_t \rightarrow 1$, and (16) imply that a stable labor share equilibrium exists only if

$$\lim (g_K - g_L) = \frac{\theta}{1-\sigma}. \quad (\text{A-3})$$

Proof of Lemma 1: From the decomposition $Y_t = K_t F_{K,t} + L_t F_{L,t}$:

$$\dot{Y}_t = \dot{K}_t F_{K,t} + K_t \dot{F}_{K,t} + \dot{L}_t F_{L,t} + L_t \dot{F}_{L,t}$$

and hence (using also (10) and (11))

$$\begin{aligned} \frac{\dot{Y}_t}{Y_t} &= \frac{\dot{K}_t}{K_t} \frac{K_t F_{K,t}}{Y_t} + \frac{K_t F_{K,t}}{Y_t} \frac{\dot{F}_{K,t}}{F_{K,t}} + \frac{\dot{L}_t}{L_t} \frac{L_t F_{L,t}}{Y_t} + \frac{L_t F_{L,t}}{Y_t} \frac{\dot{F}_{L,t}}{F_{L,t}} \\ &= \left(\frac{\sigma-1}{\sigma} \right) \left((1-X_t) \frac{\dot{K}_t}{K_t} + X_t \frac{\dot{L}_t}{L_t} \right) + \frac{1}{\sigma} \frac{\dot{Y}_t}{Y_t} + \frac{\theta}{\sigma} \left((1-X_t) \frac{1-\alpha_t}{\alpha_t} - X_t \right), \end{aligned}$$

i.e.,

$$g_{Y,t} = (1-X_t) g_{K,t} + X_t g_{L,t} + \frac{\theta}{\sigma-1} \left((1-X_t) \frac{1-\alpha_t}{\alpha_t} - X_t \right),$$

which yields the result and completes the proof.

Proof of Lemma 2: The intratemporal optimality condition (21) rewrites as $L_{i,t} \geq 1 - \omega^\eta W_t^{-\eta} C_{i,t}$, with equality if $L_{i,t} > 0$. Hence $L_{i,t}$ is increasing in wages W_t . Wages W_t are decreasing in each of $L_{o,t}, L_{w,t}$. Hence $L_{i,t}$ is decreasing in $L_{-i,t}$ (for $i = o, w$), and accordingly, $L_{o,t}$ and $L_{w,t}$ are uniquely determined by $(K_{o,t}, K_{w,t}, C_{o,t}, C_{w,t})$. Given $L_{o,t}, L_{w,t}$ and $K_{o,t}, K_{w,t}$, the rental rate R_t and wages W_t are immediate from (6) and (7), completing the proof.

Proof of Lemma 3: The proof is straightforward but tedious, and is relegated to Appendix F.

Proof of Lemma 4: First, there cannot be an equilibrium in which some group i both holds capital and has $\lim g_{1-L_i} < 0$, as follows. Suppose to the contrary that such an equilibrium exists. From the law of motion for capital,

$$\lim g_{K,i} = \bar{F}_K - \delta_i + \lim \frac{L_i F_L - C_i}{K_i}.$$

Since group i holds capital, its transversality condition can hold only if the final term on the RHS is non-positive, which in turn requires

$$\lim g_{C,i} \geq \lim g_{L,i} + \lim g_W = \lim g_W,$$

where the equality follows from the supposition that $\lim g_{1-L_i} < 0$. But since group i is not at the no-work corner, (23) and $\eta < 1$ imply that

$$\lim g_{C,i} = \eta \lim g_W + \lim g_{1-L_i} < \lim g_W,$$

contradicting the previous inequality and establishing the claim.

Second, there cannot be an equilibrium in which some group i does not hold capital and has $\lim g_{1-L_i} < 0$, as follows. Suppose to the contrary that such an equilibrium exists. Since by supposition $\lim g_{L_i} = 0$, the budget constraint for this non-capital-holding group i gives

$$\lim g_{C_i} = \lim g_W.$$

Substitution into (23) gives

$$\lim g_{1-L_i} = (1 - \eta) \lim g_W,$$

and hence (by supposition, and $\eta < 1$)

$$\lim g_W = \lim g_{C_i} \leq 0.$$

From (24),

$$\frac{\partial}{\partial t} \ln MU_{C_i} \geq 0.$$

From (13), $\lim X = 0$, and hence $\bar{F}_K = A_K$. Hence intertemporal optimality (22) and assumption (8) imply

$$\lim \frac{\partial}{\partial t} \ln MU_{C_i,t} \leq -(A_K - \delta_i - \rho) < 0.$$

The contradiction establishes the claim.

So far, we have established that $\lim g_{1-L_i} = 0$ for both groups. We now show that

$$\lim g_W > 0.$$

At least one group i must work since output is $F \equiv 0$ if $L = 0$, and the intratemporal optimality condition (23) for this group gives

$$\lim g_{C_i} = \eta \lim g_W.$$

Suppose to the contrary that $\lim g_W \leq 0$. Then one obtains a contradiction exactly as above.

The consumption of both groups grows without bound, as follows. From the previous step, wages grow without bound; and also as above, at least one group must work. The combination of that group's intratemporal condition (23) and $\lim g_{1-L_i} = 0$ implies that the consumption of any group that works grows without bound. Moreover, if a group does not work, consumption of that group must grow even faster, completing the proof.

Proof of Proposition 1:

Part (I): We first show that workers supply strictly positive labor asymptotically. To see this, suppose to the contrary that workers do not work asymptotically. Hence workers hold capital, and capitalists work. From intertemporal optimality (28),

$$\lim g_{C_w} = \eta (\bar{F}_K - \delta_w - \rho) < \eta (\bar{F}_K - \delta_o - \rho) \leq \lim g_{C_o}, \quad (\text{A-4})$$

implying that workers work (since capitalists do), contradicting the original supposition.

Similarly, capitalists hold capital asymptotically. To see this, suppose to the contrary that

capitalists do not hold capital. Hence workers hold capital, and capitalists work. Exactly the same steps as above imply (A-4), which contradicts the following implication of intratemporal optimality conditions:

$$\frac{1}{\eta} \lim g_{C_o} = \lim g_W \leq \frac{1}{\eta} \lim g_{C_w}.$$

Next, we show that capitalists do not work under capital dominance. Suppose to the contrary that capitalists and workers both work. By Corollary 1, workers do not hold capital. By capital dominance, aggregate labor income grows strictly slower than $\lim g_Y = \lim g_K$, and hence workers' consumption C_w likewise grows strictly slower than $\lim g_K$. Capitalists' capital accumulation is given by

$$\frac{\dot{K}_{o,t}}{K_{o,t}} = F_{K,t} - \delta_o + \frac{L_{o,t} F_{L,t}}{Y_t} \frac{Y_t}{K_{o,t}} - \frac{C_{o,t}}{K_{o,t}}.$$

By capital dominance, the third term on the RHS converges to 0. The transversality condition for capitalists then implies that capitalists' consumption C_o asymptotically grows at the same rate as their capital holdings K_o , i.e.,

$$\lim g_{C_o} = \lim g_{K_o} = \lim g_K.$$

Hence capitalists' consumption grows strictly faster than workers' consumption, and the intratemporal optimality conditions imply that capitalists do not work, contradicting the supposition that they do.

Finally, we show that workers do not hold capital in stable labor share equilibrium. Suppose to the contrary that both capitalists and workers hold capital. (28) at equality for both groups directly implies $\lim g_{C_o} > \lim g_{C_w}$. Moreover, from Corollary 1, capitalists do not work, and the transversality condition for capitalists implies $\lim g_{K_o} = \lim g_{C_o}$. Workers' capital accumulation is given by

$$g_{K_w,t} = \frac{\dot{K}_{w,t}}{K_{w,t}} = F_{K,t} - \delta_w + \frac{L_{w,t} F_{L,t}}{Y_t} \frac{Y_t}{K_{w,t}} - \frac{C_{w,t}}{K_{w,t}}.$$

If $\lim g_{K_w} \geq \lim g_{K_o}$ then

$$\lim g_Y = \lim g_K = \lim g_{K_w} \geq \lim g_{K_o} = \lim g_{C_o} > \lim g_{C_w}$$

implying

$$\lim g_{K_w} = \bar{F}_K - \delta_w + \lim X \lim \frac{Y}{K} > \bar{F}_K - \delta_w,$$

violating the workers' transversality condition. If instead $\lim g_{K_w} < \lim g_{K_o}$ then

$$\lim g_Y = \lim g_K = \lim g_{K_o} = \lim g_{C_o} > \lim g_{C_w},$$

implying that $\frac{L_{w,t}F_{L,t}}{Y_t}Y_t - C_{w,t}$ asymptotically grows at the same rate as aggregate capital K , which strictly exceeds the growth rate of worker capital K_w , implying $\lim g_{K_w} > \bar{F}_K - \delta_w$ and violating the workers' transversality condition. The contradiction completes the proof of Part (I).

Part (II): Immediate from Part (I)'s finding that workers supply labor, together with the intratemporal optimality condition (21).

Part (III): Recall that $\lim g_Y = \lim g_K$ (see (16)). From Part (II), the asymptotic growth rate of capitalists' consumption coincides with the asymptotic growth rate of aggregate consumption, $\lim g_{C_o} = \lim g_C$. Asymptotically, aggregate consumption must grow weakly slower than output, $\lim g_C \leq \lim g_Y$. For both groups i , the asymptotic growth rate of capital must be weakly below the asymptotic growth rate of consumption, $\lim g_{K_i} \leq \lim g_{C_i}$, since otherwise that group's transversality condition is violated.

We next show that $\lim g_{K_o} = \lim g_K$. If workers do not hold capital then this is immediate. If workers do hold capital, it suffices to show that $\lim g_{K_o} \geq \lim g_{K_w}$. Suppose to the contrary that $\lim g_{K_o} < \lim g_{K_w}$. In this case, capitalists do not work, and since the return on capital asymptotes to \bar{F}_K , capitalists' consumption must grow weakly slower than capitalists' capital holdings, $\lim g_{C_o} \leq \lim g_{K_o}$. Together, the above inequalities deliver

$$\lim g_{C_o} \leq \lim g_{K_o} < \lim g_{K_w} \leq \lim g_{C_w},$$

contradicting Part (II), and thereby establishing that $\lim g_{K_o} = \lim g_K$.

To complete the proof of Part (III), simply note that

$$\lim g_{C_o} = \lim g_C \leq \lim g_Y = \lim g_K = \lim g_{K_o} \leq \lim g_{C_o}.$$

establishing the result.

Proof of Proposition 2: We characterize the conditions for a capital-dominant equilibrium in which both groups hold capital to exist. From Proposition 1, workers work while capitalists do not. In a capital-dominant equilibrium, $\bar{F}_K = A_K$, and so from (28), the intertemporal conditions for capitalists and workers are

$$\begin{aligned}\lim g_{C_o} &= \eta(A_K - \delta_o - \rho) \\ \lim g_{C_w} &= \eta(A_K - \delta_w - \rho)\end{aligned}$$

while the intratemporal condition for workers is (using Lemma 4)

$$\lim g_W = \frac{1}{\eta} \lim g_{C_w} = A_K - \delta_w - \rho.$$

(Note that the above expression is positive by assumption (8).) Capital holdings grow according to

$$\begin{aligned}\lim g_{K_o} &= A_K - \delta_o - \lim \frac{C_o}{K_o} \\ \lim g_{K_w} &= A_K - \delta_w + \lim \frac{L_w F_L - C_w}{K_w},\end{aligned}$$

and from (A-2), wages grow according to

$$\lim g_W = \frac{1}{\sigma} (\lim g_K - \lim g_{L_w} - \theta).$$

Capitalists' transversality condition implies that C_o and K_o asymptotically grow at the same rate:

$$\lim g_{K_o} = \lim g_{C_o} = \eta(A_K - \delta_o - \rho).$$

We characterize an equilibrium in which C_w and K_w asymptotically grow at the same rate. In this case,

$$\lim g_{K_w} < \lim g_{K_o} = \lim g_K,$$

and so

$$\lim g_{L_w} = \eta(A_K - \delta_o - \rho) - \sigma(A_K - \delta_w - \rho) - \theta.$$

A worker's transversality condition is equivalent to

$$\lim g_{C_w} \geq \lim g_W + \lim g_{L_w}, \tag{A-5}$$

which substituting in the above expressions is equivalent to

$$\eta(A_K - \delta_w - \rho) \geq A_K - \delta_w - \rho + \eta(A_K - \delta_o - \rho) - \sigma(A_K - \delta_w - \rho) - \theta,$$

and hence to

$$\theta \geq (1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o). \tag{A-6}$$

Note that $\lim g_{C_o} > \lim g_{C_w}$ together with the worker transversality condition (A-5) implies that the capital-dominance condition is satisfied; and also that capitalists indeed do not work. Moreover, the worker transversality condition implies that $\lim g_{L_w} < 0$.

Proof of Proposition 3: We characterize the conditions for a capital-dominant equilibrium in which workers do not hold capital to exist. By similar arguments to those in the proof of Proposition 2, the asymptotic equilibrium conditions are as follows. (Relative to the proof of Lemma 2, the key difference is that workers' intertemporal optimality condition is replaced with an intratemporal budget constraint.)

$$\begin{aligned} \lim g_{K_o} = \lim g_{C_o} &= \eta(A_K - \delta_o - \rho) \\ \lim g_W &= \frac{1}{\eta} \lim g_{C_w} \\ \lim g_{C_w} &= \lim g_W + \lim g_{L_w} \\ \lim g_W &= \frac{1}{\sigma} (\lim g_{K_o} - \lim g_{L_w} - \theta). \end{aligned}$$

From a worker's intratemporal optimality and intratemporal budget constraint,

$$\lim g_{L_w} = (\eta - 1) \lim g_W.$$

Hence

$$\lim g_W = \frac{\lim g_{K_o} - \theta}{\sigma + \eta - 1}.$$

The capital-dominance condition is $\lim g_{K_o} > \lim g_W + \lim g_{L_w}$. Note that if the capital-dominance condition holds then $\lim g_{C_o} > \lim g_{C_w}$, which ensures that capitalists indeed do not work asymptotically. Substituting in, the capital-dominance condition is

$$\lim g_{K_o} > \eta \frac{\lim g_{K_o} - \theta}{\sigma + \eta - 1}.$$

The condition that workers asymptotically do not want to hold capital is (from (28), and substituting in for $\lim g_{C_w}$)

$$\lim g_W \geq A_K - \delta_w - \rho,$$

i.e.,

$$\lim g_W = \frac{\lim g_{K_o} - \theta}{\sigma + \eta - 1} \geq A_K - \delta_w - \rho = \frac{1}{\eta} \lim g_{K_o} - (\delta_w - \delta_o).$$

The above condition and (8) imply that $\lim g_W > 0$ and $\lim g_{L_w} < 0$.

Hence an equilibrium of this type exists if either $\sigma + \eta > 1$ and

$$\theta \in \left[\frac{1 - \sigma}{\eta} \lim g_{K_o}, \frac{1 - \sigma}{\eta} \lim g_{K_o} + (\sigma + \eta - 1) (\delta_w - \delta_o) \right]$$

or if $\sigma + \eta < 1$ and

$$\theta \in \left[\frac{1 - \sigma}{\eta} \lim g_{K_o} + (\sigma + \eta - 1) (\delta_w - \delta_o), \frac{1 - \sigma}{\eta} \lim g_{K_o} \right]$$

Substituting in for $\lim g_{K_o}$ yields the result.

Finally, we show that an equilibrium of this type is stable if and only if complementarities are weak. Because workers don't hold capital for t large enough, $C_{w,t} = W_t L_{w,t}$, and so intratemporal

optimality is

$$-\frac{1}{\eta} \log(1 - L_{w,t}) = -\frac{1}{\eta} ((1 - \eta) \log W_t + \log L_{w,t}) - \log \omega. \quad (\text{A-7})$$

The LHS of this equality is proportional to the log of marginal utility of leisure, and is increasing in $L_{w,t}$. The RHS of this equality is proportional to the log of the product of W_t and the marginal utility of consumption; its derivative with respect to $L_{w,t}$ is

$$-\frac{1}{\eta} \left(\frac{1}{L_{w,t}} - \frac{1 - \eta}{\sigma} \left(\frac{1}{L_{w,t}} - \frac{W_t}{Y_t} \right) \right) = -\frac{1}{\eta} \frac{1}{L_{w,t}} \left(1 - \frac{1 - \eta}{\sigma} (1 - X_t) \right).$$

In any equilibrium $L_{w,t} \rightarrow 0$ and in a capital-dominant equilibrium, $X_t \rightarrow 0$. So under strong complementarities ($\frac{1-\eta}{\sigma} > 1$) the slope of the RHS of (A-7) grows arbitrarily large at the equilibrium point, implying the RHS crosses the LHS from below. Consequently, the equilibrium is unstable: an upwards perturbation to $L_{w,t}$ results in $W_t MU_{C_{w,t}} > MU_{1-L_{w,t}}$, implying that workers should further increase $L_{w,t}$. This completes the proof.

Proof of Corollary 2: To see why the return on capital in a capital-dominant equilibrium exceeds today's rate, first note that substitutability of capital and labor in automated tasks implies that, in equilibrium,

$$\frac{A_K}{F_{K,t}} \geq \frac{A_L}{F_{L,t}},$$

or equivalently,

$$\frac{A_K K_t}{\tilde{\alpha}_t} \geq \frac{A_L L_t}{1 - \tilde{\alpha}_t},$$

where $\tilde{\alpha}_t \leq \alpha_t$ is the fraction of tasks that are actually automated (i.e., tasks for which automation is feasible, and for which labor is replaced with capital; as noted, asymptotically capital is abundant, and so labor is replaced by capital whenever feasible). It follows that

$$Y_t \leq \frac{A_K K_t}{\tilde{\alpha}_t}$$

and hence, using (5),

$$F_{K,t} \leq A_K.$$

Moreover, the inequality is strict provided that it is strictly profitable to replace labor with capital.

Proof of Proposition 4: We characterize the conditions for a stable labor share equilibrium to exist. From Proposition 1, workers do not hold capital. Following similar steps to those in the proofs of Propositions 2 and 3, but incorporating the possibility that capitalists work, the asymptotic equilibrium conditions are

$$\begin{aligned}
\lim g_{C_o} &\geq \eta \lim g_W \\
\lim g_{C_o} &= \eta (\bar{F}_K - \delta_o - \rho) \\
\lim g_{K_o} &= \bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o} \\
\lim g_W &= \frac{1}{\eta} \lim g_{C_w} \\
\lim g_{C_w} &= \lim g_W + \lim g_{L_w} \\
\lim g_W &= \frac{\theta}{1 - \sigma}.
\end{aligned}$$

From Part (III) of Proposition 1,

$$\lim g_Y = \lim g_{K_o} = \lim g_{C_o} = \eta (\bar{F}_K - \delta_o - \rho).$$

We first show that aggregate labor growth matches worker-labor growth, i.e.,

$$\lim g_L = \lim g_{L_w}. \tag{A-8}$$

If capitalists do not work then (A-8) is immediate. If instead capitalists work, note that capital evolves according to

$$\lim g_{K_o} = \bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o}.$$

Capitalists' transversality constraint implies that their labor income grows weakly slower than the common growth rate of their consumption and capital. Moreover, if both capitalists and workers work, their consumption growth rates must asymptotically coincide (by Lemma 4 and the intratem-

poral optimality conditions). Hence

$$\lim g_W + \lim g_{L_o} \leq \lim g_{C_o} = \lim g_{C_w} = \lim g_W + g_{L_w}, \quad (\text{A-9})$$

implying that $\lim g_{L_o} \leq \lim g_{L_w}$ and establishing (A-8).

From the workers' intratemporal optimality and intratemporal budget constraint,

$$\lim g_{L_w} = (\eta - 1) \lim g_W = (\eta - 1) \frac{\theta}{1 - \sigma}.$$

Note that this condition ensures that $\lim g_{L_w} < 0$. Further, from (A-3), a stable labor share requires

$$\lim g_{K_o} - \lim g_L = \frac{\theta}{1 - \sigma}.$$

From (A-8), it follows that

$$\lim g_{K_o} = \eta \frac{\theta}{1 - \sigma},$$

which combined with capitalists' intertemporal optimality implies that the limiting rental rate is

$$\bar{F}_K = \frac{\theta}{1 - \sigma} + \delta_o + \rho. \quad (\text{A-10})$$

From (17), the asymptotic capital share is bounded away from one if and only if $\left(\frac{\bar{F}_K}{A_K}\right)^{1-\sigma} < 1$, which after substitution for \bar{F}_K is equivalent to

$$\frac{\theta}{1 - \sigma} + \delta_o + \rho < A_K.$$

Rearranging establishes the stable labor share condition, (35).

Workers' and capitalists' consumption grow at the same asymptotic rate, as follows. If capitalists do not work, this is immediate from the combination of definition of a stable labor share and the fact that output F , capital K_o and capitalist consumption C_o all grow at the same rate. If instead capitalists work, then it follows intratemporal optimality conditions, as already noted in (A-9).

Finally, the expression for the limiting labor share follows from the substitution of \bar{F}_K into (17).

This completes the proof.

Proof of Lemma 5: Multiplicity in steady states arises when complementarities are strong ($\sigma + \eta < 1$) and

$$\theta \in [(1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o), (1 - \sigma)(A_K - \delta_o - \rho)]. \quad (\text{A-11})$$

In the stable labor share equilibrium,

$$\begin{aligned} \lim g_{C_o} = \lim g_{C_w} &= \frac{\eta}{1 - \sigma} \theta \\ \lim g_{L_w} &= \frac{\eta - 1}{1 - \sigma} \theta \end{aligned}$$

while in the capital-dominant equilibrium,

$$\begin{aligned} \lim g_{C_o} &= \eta(A_K - \delta_o - \rho) \\ \lim g_{C_w} &= \eta(A_K - \delta_w - \rho) \\ \lim g_{L_w} &= \eta(A_K - \delta_o - \rho) - \sigma(A_K - \delta_w - \rho) - \theta. \end{aligned}$$

It is immediate that $\lim g_{C_o}$ (respectively, $\lim g_{C_w}$) is higher (lower) in the capital-dominant equilibrium than in the stable labor share equilibrium. Moreover, both comparisons are strict, with the exception of $\lim g_{C_o}$ at the upper boundary of the interval (A-11).

It remains to consider the labor growth rate $\lim g_{L_w}$. Because it is linear in θ in both equilibria, it suffices to consider the lower and upper boundaries of the interval (A-11).

At the lower end of the interval, in the stable labor share equilibrium

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_w - \rho) + \frac{\eta(\eta - 1)}{1 - \sigma}(\delta_w - \delta_o),$$

while in the capital-dominant equilibrium,

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_w - \rho),$$

which is strictly greater.

At the upper end of the interval, in the stable labor equilibrium

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_o - \rho),$$

while in the capital-dominant equilibrium,

$$\lim g_{L_w} = (\eta - 1)(A_K - \delta_o - \rho) + \sigma(\delta_w - \delta_o).$$

which again is strictly greater, completing the proof.

Proof of Proposition 5: We exogenously set labor choices to $L_{o,t} = 0$ and $g_{L_w} < 0$.²⁰ Intertemporal optimality of capitalists implies

$$\lim g_{K_o} = \eta(\bar{F}_K - \delta_o - \rho). \tag{A-12}$$

As before, aggregate capital growth equals capitalists' capital growth,

$$\lim g_K = \lim g_{K_o},$$

regardless of whether or not workers hold capital (since even if workers hold capital, their capital holdings grow more slowly than that of capitalists).

As before, the condition for capital dominance is that labor income asymptotically grows slower than capital income, i.e.,

$$g_{L_w} + \lim g_W < \lim g_K.$$

From (A-2), capital dominance also requires

$$\lim g_W = \frac{1}{\sigma}(\lim g_K - g_{L_w} - \theta).$$

²⁰Note that we do not remove leisure from the agents' preferences. In the limit as consumption grows unbounded but leisure is bounded, the IES with consumption and leisure as gross complements tends to η rather than $1/\gamma$. Retaining this feature of preferences in the exogenous-labor case facilitates comparison with the endogenous-labor case.

Finally, under capital dominance the return on capital asymptotes to A_K . Together, these observations imply that capital dominance requires

$$\theta > (1 - \sigma) (\eta (A_K - \delta_o - \rho) - g_{L_w}). \quad (\text{A-13})$$

When (A-13) is satisfied,

$$\lim g_Y = \lim g_{K_o} = \lim g_{C_o} = \eta (A_K - \delta_o - \rho). \quad (\text{A-14})$$

Workers' consumption grows at rate

$$\max \left\{ \left(1 - \frac{1}{\sigma} \right) g_{L_w} + \frac{\eta}{\sigma} (A_K - \delta_o - \rho) - \frac{\theta}{\sigma}, \eta (A_K - \delta_w - \rho) \right\}, \quad (\text{A-15})$$

where the two terms in the maximand correspond, respectively, to workers not holding capital, and holding capital. Note that if (A-13) holds at equality then worker consumption growth (A-15) coincides with (A-14).

Conversely, in a stable labor share equilibrium, consumption, capital income, and labor income must grow at the same rate. Equation (14) characterizes wage growth in a stable labor share equilibrium. Equation (A-12) gives the growth rate of capital. Hence a stable labor equilibrium requires

$$\frac{\theta}{1 - \sigma} + g_{L_w} = \eta (\bar{F}_K - \delta_o - \rho).$$

As in the proof of Proposition 4, a stable labor share equilibrium requires $\bar{F}_K < A_K$, and hence requires

$$\theta < (1 - \sigma) (\eta (A_K - \delta_o - \rho) - g_{L_w}),$$

which combined with (A-13) establishes the threshold rate of automation that determines a stable labor share vs capital dominance.

From (A-3), in a stable labor share equilibrium

$$\lim g_K = \frac{\theta}{1 - \sigma} + g_{L_w}.$$

Combining with (A-12) yields

$$\eta (\bar{F}_K - \delta_o - \rho) = \frac{\theta}{1 - \sigma} + g_{L_w},$$

thereby characterizing \bar{F}_K . Given \bar{F}_K , the growth rate of the economy is given by (A-12). The asymptotic capital share is then determined by (17), completing the proof.

Proof of Proposition 6: First note that the capital-dominant steady state in which workers hold capital no longer exists since, in this case, worker and capitalist consumption growth would coincide. Then $\delta_w > \delta_o$ implies that the intertemporal optimality conditions (22) cannot hold for both workers and capitalists.

Next, we show that there is also no capital-dominant steady state in which workers do not hold capital, while the stable labor share steady state remains. By similar arguments to those in the proof of Propositions 2 and 3, the asymptotic equilibrium conditions are as follows. (Relative to the proof of Proposition 3, the key difference is that wage growth takes the policy-imposed value, equal to its equilibrium value in the stable labor share steady state.)

$$\begin{aligned} \lim g_{K_o} = \lim g_{C_o} &= \eta (F_K - \delta_o - \rho) \\ \lim g_W &= \frac{1}{\eta} \lim g_{C_w} = \frac{\theta}{1 - \sigma} \\ \lim g_{C_w} &= \lim g_W + \lim g_{L_w} \\ \lim g_W &= \frac{1}{\sigma} (\lim g_{K_o} - \lim g_{L_w} - \theta). \end{aligned}$$

From a worker's intratemporal optimality and intratemporal budget constraint,

$$\lim g_{L_w} = (\eta - 1) \frac{\theta}{1 - \sigma}$$

Substituting into the equilibrium wage growth expression:

$$\frac{\theta}{1 - \sigma} = \frac{1}{\sigma} \left(\eta (F_K - \delta_o - \rho) - (\eta - 1) \frac{\theta}{1 - \sigma} - \theta \right)$$

and hence

$$\theta = (1 - \sigma)(F_K - \delta_o - \rho) \leq (1 - \sigma)(A_K - \delta_o - \rho).$$

Since this is the capital-dominance threshold, there is no capital-dominant equilibrium.

To see why the stable labor share steady state described in Subsection 2.5 still exists, simply note that the described policy does not alter any of its conditions: workers' equilibrium consumption growth is equal to the policy's guarantee. Capital holdings are also unchanged, workers do not hold capital in this steady state and capitalists' savings are unchanged given that the equilibrium tax rate $\tau = 0$ from (44).

Output and consumption grow at

$$g_Y^{SLS} = g_{C_o}^{SLS} = g_{C_w}^{SLS} = \frac{\eta\theta}{1 - \sigma}.$$

This is lower than total output growth in the capital-dominant steady state which equals capitalists' consumption growth and, by (31), is given by

$$g_Y^{CD} = g_{C_o}^{CD} = \eta(A_K - \delta_o - \rho) > g_Y^{SLS},$$

where the inequality follows from (43), completing the proof.

Proof of Proposition 7: We assume throughout that complementarities are strong, $\sigma + \eta < 1$, since otherwise multiplicity does not arise in the baseline case of no labor taxation.

Suppose labor retains a fraction λ_t of the wage W_t . From intratemporal optimality (23), workers supply labor only if

$$\lim g_{C_w} = \eta(\lim g_W + \lim g_\lambda). \tag{A-16}$$

Combining with (11), the growth rate of labor income satisfies

$$\lim g_W + \lim g_{L_w} = \lim g_{K_o} + (1 - \sigma) \left(\frac{\lim g_{C_w}}{\eta} - \lim g_\lambda \right) - \theta. \tag{A-17}$$

Intertemporal optimality for any agent holding capital is given by (28). Transversality further

implies that

$$\lim g_{C_o} = \lim g_{K_o} \tag{A-18}$$

$$\lim g_{C_w} \geq \lim g_W + \lim g_{L_w} + \lim g_\lambda. \tag{A-19}$$

We now characterize the necessary conditions under for each of the two originally coexisting steady states. First, for capital dominance, worker transversality (A-19) and labor income growth (A-17) imply

$$\theta \geq \lim g_{K_o} + \left(\frac{1-\sigma}{\eta} - 1 \right) \lim g_{C_w} + \sigma \lim g_\lambda.$$

Since complementarities are strong, $\frac{1-\sigma}{\eta} > 1$, and capitalist transversality (A-18) and intertemporal optimality (28) imply

$$\theta \geq (1-\sigma)(\bar{F}_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) + \sigma g_\lambda.$$

Since capital dominance implies $\bar{F}_K = A_K$, capital dominance can arise only if

$$\theta \geq (1-\sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) + \sigma \lim g_\lambda, \tag{A-20}$$

meaning an ever increasing labor tax ($\lim g_\lambda < 0$) lowers the capital-dominance threshold.

Second, we characterize the necessary condition for the stable (pre-tax) labor share steady state to exist. This requires

$$\lim g_{K_o} = \lim g_W + \lim g_{L_w}. \tag{A-21}$$

Combining with (A-17), this implies

$$\theta = \frac{1-\sigma}{\eta} \lim g_{C_w} - (1-\sigma) \lim g_\lambda.$$

In the original stable labor share steady state, workers are hand-to-mouth.²¹ Combining the budget

²¹If workers instead hold capital, then $g_{C_w} = \eta(R - \delta_w - \rho)$. This is greater than the growth of post-tax labor income if and only if $g_\lambda < -\eta(\delta_w - \delta_o)$. Looking ahead, this means that workers hold capital only for levels of g_λ that eliminate the multiplicity region and, thus, the stable labor share steady state under consideration given θ in the undistorted multiplicity region.

constraint with (A-21) and the capitalists' Euler equation (28) gives

$$\lim g_{C_w} = \eta(\bar{F}_k - \delta_o - \rho) + \lim g_\lambda$$

Substituting g_{C_w} from above yields

$$\theta = (1 - \sigma)(\bar{F}_K - \delta_o - \rho) + \frac{1 - \sigma}{\eta}(1 - \eta) \lim g_\lambda,$$

which implies that a stable labor share steady state exists only if

$$\theta \leq (1 - \sigma)(A_K - \delta_o - \rho) + \frac{1 - \sigma}{\eta}(1 - \eta) \lim g_\lambda. \quad (\text{A-22})$$

Combining with (A-20), the two steady states coexist for θ in range

$$(1 - \sigma)(A_K - \delta_w - \rho) + \eta(\delta_w - \delta_o) + \sigma \lim g_\lambda \leq \theta \leq (1 - \sigma)(A_K - \delta_o - \rho) + \frac{1 - \sigma}{\eta}(1 - \eta) \lim g_\lambda.$$

Rearranging, this range shrinks to zero for $\lim g_\lambda \leq -\eta(\delta_w - \delta_o)$. If θ was in the original multiplicity range (43), this tax rate eliminates the stable labor share steady state. In the resultant capital-dominant steady state, labor income as a fraction of output asymptotes to zero as does, therefore, the deadweight cost of the labor tax. The output growth rate in this capital-dominant steady state is the same as in the undistorted capital-dominant steady state, that is, $\lim g_Y = \eta(A_K - \delta_o - \rho)$. This completes the proof.

Proof of Lemma 6: From intratemporal optimality (21) (and dropping all time and agent subscripts for clarity), if an agent works then

$$\Delta \ln W - \frac{1}{\eta} \Delta \ln C = -\frac{1}{\eta} \Delta \ln(1 - L),$$

or equivalently,

$$\frac{L}{1 - L} \Delta \ln L = \eta \Delta \ln W - \Delta \ln C. \quad (\text{A-23})$$

The Frisch elasticity holds the marginal utility of capital constant—and hence holds the marginal

utility of consumption constant. From (19), and after factoring out common terms, a constant marginal utility of consumption is equivalent to

$$\begin{aligned} & \left(-\frac{1}{\eta}C^{-1} + \frac{1-\eta\gamma}{\eta}C^{-\frac{1}{\eta}} \left(C^{\frac{\eta-1}{\eta}} + \omega(1-L)^{\frac{\eta-1}{\eta}} \right)^{-1} \right) \Delta C \\ &= \left(\frac{1-\eta\gamma}{\eta}\omega(1-L)^{\frac{-1}{\eta}} \left(C^{\frac{\eta-1}{\eta}} + \omega(1-L)^{\frac{\eta-1}{\eta}} \right)^{-1} \right) \Delta L. \end{aligned}$$

Substituting in the intratemporal optimality condition (21), and multiplying by $\frac{\eta}{\gamma}$,

$$\begin{aligned} & \left(-\frac{1}{\gamma}C^{-1} + \left(\frac{1}{\gamma} - \eta \right) C^{-\frac{1}{\eta}} \left(CC^{\frac{-1}{\eta}} + W(1-L)C^{\frac{-1}{\eta}} \right)^{-1} \right) \Delta C \\ &= \left(\left(\frac{1}{\gamma} - \eta \right) WC^{-\frac{1}{\eta}} \left(CC^{\frac{-1}{\eta}} + W(1-L)C^{\frac{-1}{\eta}} \right)^{-1} \right) \Delta L, \end{aligned}$$

which simplifies to

$$\left(-\frac{1}{\gamma} \frac{C + W(1-L)}{C} + \frac{1}{\gamma} - \eta \right) \Delta C = \left(\frac{1}{\gamma} - \eta \right) W \Delta L,$$

and hence to

$$\left(-\frac{1}{\gamma}W(1-L) - \eta C \right) \Delta \ln C = \left(\frac{1}{\gamma} - \eta \right) WL \Delta \ln L,$$

and in turn to

$$\Delta \ln C = \frac{\left(\eta - \frac{1}{\gamma} \right) \frac{WL}{C}}{\eta + \frac{1}{\gamma} \frac{1-L}{L} \frac{WL}{C}} \Delta \ln L.$$

Substituting into (A-23) gives

$$\frac{\eta \frac{L}{1-L} + \eta \frac{WL}{C}}{\eta + \frac{1}{\gamma} \frac{1-L}{L} \frac{WL}{C}} \Delta \ln L = \eta \Delta W,$$

and hence the following expression for the Frisch elasticity of labor supply,

$$\frac{\Delta \ln L}{\Delta \ln W} = \frac{\eta + \frac{1}{\gamma} \frac{1-L}{L} \frac{WL}{C}}{\frac{WL}{C} + \frac{L}{1-L}}.$$

Rearranging delivers (59).

Finally, the intratemporal optimality (21) condition rearranges as

$$\frac{WL}{C} \frac{1-L}{L} C^{\frac{\eta-1}{\eta}} = \omega (1-L)^{\frac{\eta-1}{\eta}},$$

and substitution into (19) yields

$$MU_C = C^{-\gamma} \left(1 + \frac{1-L}{L} \frac{WL}{C} \right)^{\frac{1-\eta\gamma}{\eta-1}}.$$

So provided that the IES is estimated over periods in which $1 + \frac{1-L}{L} \frac{WL}{C}$ doesn't change too much,²²

$$IES \approx \frac{1}{\gamma},$$

completing the proof.

²²To relate this statement to the observation in the main text that the IES asymptotes to η on the equilibrium path: From intratemporal optimality, $C = \left(\frac{W}{\omega}\right)^\eta (1-L)$. So $MU_C = \left(\frac{W}{\omega}\right)^{-\eta\gamma} (1-L)^{-\gamma} \left(1 + \omega^\eta W^{1-\eta}\right)^{\frac{1-\eta\gamma}{\eta-1}}$. As W grows large and L approaches 0, MU_C becomes proportional to W^{-1} . Hence the IES approaches η .

B Calibration

B.1 Details for inputs

Table 1 below reports the inputs we use in our calibration, along with sources.^{23,24}

Input	Description	Value	Source
g_Y	Growth rate of output	2.79%	National income accounts
g_L	Growth rate of labor (per worker)	-0.19%	National income accounts
g_K	Growth rate of capital	2.42%	National income accounts
g_X	Growth rate of labor share	-0.17%	National income accounts
g_{1-X}	Growth rate of capital share	0.28%	National income accounts
g_C	Growth rate of consumption (per capita)	2.00%	National income accounts
X	Labor share	59.7%	National income accounts
$\frac{K_t}{Y_t}$	Capital/output ratio	3.63	National income accounts
δ_o	Depreciation	4.32%	National income accounts
ρ	Annual time preference	2%	Standard
	Employment growth rate	1.43%	National income accounts

Table 1: Input values. Levels from national income accounts are estimated as of 2019. Growth rates refer to relative changes between 1970 and 2019. The growth rates of output and capital are for aggregate quantities; the calibration uses per-capita growth (subtracting population growth).

B.2 Alternative calibration approaches

We pursue three alternative calibration approaches complementing the exercise in Section 6.²⁵

First, and as mentioned in the main text, we start from Acemoglu (2024)’s estimate that 5% of GDP will be disrupted by AI in the next 10 years. In our notation (writing t for the current time period), tasks $\alpha_{t+10} - \alpha_t$ will be automated over the next 10 years, and the current GDP share of these tasks is

$$(\alpha_{t+10} - \alpha_t) \frac{X_t}{1 - \alpha_t}. \quad (\text{B-1})$$

²³The growth rates of output, labor supply, and the labor share are not constant in our model. We estimate them using data from 1970 to 2019.

²⁴The empirical measurement of depreciation corresponds to $\frac{\lambda_o K_{o,t}}{K_t} \delta_o + \frac{\lambda_w K_{w,t}}{K_t} \delta_w \geq \delta_o$. Using a smaller value of δ_o than 4.32% would increase the estimated value of the RHS of (54), and reinforce the conclusion below that empirically the condition is likely to hold.

²⁵A fourth possible approach to estimating θ would be to use the TFP equation (52). However, such an approach is susceptible to two significant pitfalls, and accordingly we do not pursue it. First, the inferred value of θ is very sensitive to the input α_t when α_t is close in value to the capital share of the economy X_t , which we cannot rule out a priori. Second, extracting θ from the TFP formula (52) is sensitive to the model assumption that technological advance consists solely of changes in the fraction of automated tasks.

The share of tasks $1 - \alpha_{t+10}$ still unautomated in 10 years is related to today's share of unautomated tasks $1 - \alpha_t$ by

$$1 - \alpha_{t+10} = (1 - \alpha_t) \exp(-10\theta),$$

which straightforwardly rewrites as

$$\exp(-10\theta) = 1 - \frac{(\alpha_{t+10} - \alpha_t) \frac{X_t}{1 - \alpha_t}}{X_t}.$$

Substituting in the current labor share of $X_t \approx 60\%$ and Acemoglu's estimate of (B-1) of $\approx 5\%$ yields

$$\theta \approx 0.9\%,$$

A second alternative approach to calibration starts with an equation analogous to (57) but obtained from the law of motion of the capital share:

$$\frac{1 - \alpha_t}{\alpha_t} \theta = \sigma g_{1-X,t} + (1 - \sigma) (g_{K,t} - g_{Y,t}). \quad (\text{B-2})$$

Given observable growth rates for the capital share, capital, and output, the LHS can be estimated directly from existing estimates of the elasticity parameter σ .

To move from (B-2) to an estimate of θ one needs information about α_t , the fraction of tasks already automated. This number is hard to observe directly, and in our approach below we are agnostic about its value.

For any given value of α_t , we can further tighten the estimate of A_K relative to the lower bound presented in (56). The expression for the capital share (17) can be rewritten to yield

$$A_K = \frac{Y_t}{K_t} \left(\frac{\alpha_t}{(1 - X_t)^\sigma} \right)^{\frac{1}{1-\sigma}}. \quad (\text{B-3})$$

The drawback of this expression, relative to the lower bound in (56) is that it requires an assumption on α_t . Note, however, that both the value of θ inferred from (B-2) and the value of A_K inferred from expression (B-3) are increasing in the automation share α_t , and hence both the estimated LHS and RHS of the key inequality (54) are likewise increasing in α_t .

Using once again inputs from Table 1, with $g_{K,t}$ and $g_{1-X,t}$ estimated over the same 1970–2019 sample as $g_{Y,t}$, Table 2 displays, for a range of possible values of α_t and σ , the rate of automation θ (calculated from (B-2)), the key ratio $\frac{\theta}{1-\sigma}$, and the productivity parameter A_K (calculated using (B-3)). The ratio $\frac{\theta}{1-\sigma}$ only exceeds this bound if the elasticity parameter σ is relatively close to 1 and the fraction of tasks already automated (α_t) is high. Note that the baseline calibration in Section 6 implies an estimate of α_t (from equations (57) and (B-2) and the observable growth rates). These estimates indicate that the majority of tasks is already automated but are *decreasing* in σ : for $\sigma = 0.8$, the implied α_t is 0.8, dropping to 0.6 for $\sigma = 0.9$.

In the table, we use color shading to highlight the combinations of σ and α_t for which the ratio $\frac{\theta}{1-\sigma}$ either exceeds 4.79%, or at least approaches it. But those parameter choices that deliver $\frac{\theta}{1-\sigma}$ anywhere close to the boundary of 4.79% *also* imply large values for A_K , and hence for $A_K - \delta_o - \rho$, so that the stable labor share inequality (54) continues to hold.²⁶

		$\sigma = 0.6$			$\sigma = 0.8$			$\sigma = 0.9$		
α_t	θ	$\theta/(1-\sigma)$	A_K	θ	$\theta/(1-\sigma)$	A_K	θ	$\theta/(1-\sigma)$	A_K	
0.1	0.00%	0.01%	0.33%	0.02%	0.09%	0.01%	0.02%	0.24%	0.00%	
0.2	0.01%	0.02%	1.89%	0.04%	0.19%	0.33%	0.05%	0.54%	0.01%	
0.3	0.01%	0.03%	5.20%	0.07%	0.33%	2.49%	0.09%	0.93%	0.57%	
0.4	0.02%	0.04%	10.68%	0.10%	0.51%	10.48%	0.15%	1.45%	10.10%	
0.5	0.02%	0.06%	18.66%	0.15%	0.77%	31.99%	0.22%	2.18%	94.04%	
0.6	0.04%	0.09%	29.43%	0.23%	1.15%	79.60%	0.33%	3.27%	582.28%	
0.7	0.06%	0.14%	43.27%	0.36%	1.79%	172.04%	0.51%	5.08%	2720.19%	
0.8	0.10%	0.24%	60.41%	0.61%	3.06%	335.42%	0.87%	8.71%	10339.94%	
0.9	0.22%	0.54%	81.10%	1.38%	6.89%	604.44%	1.96%	19.60%	33577.12%	

Table 2: $\frac{\theta}{1-\sigma}$ and A_K as functions of the current level of automation, α_t , and the elasticity of substitution between tasks, σ . The automation rate θ is inferred from (B-2). Color shading highlights values of σ and α_t for which the ratio $\frac{\theta}{1-\sigma}$ approaches or exceeds the lower bound (58).

Finally, a third independent approach to estimating $\frac{1-\alpha_t}{\alpha_t}\theta$ is as follows. The fraction of invest-

²⁶Note that high values of α_t lead to extremely high estimates of A_K , the productivity of capital in an all-capital economy. The reason is as follows. First, note from (17) that the capital share is decreasing in the amount of “effective” capital $A_K K_t$, since tasks are complements ($\sigma < 1$). The current capital share in the economy is much less than 100%. If one believes that most tasks are already automated, the only way to explain the observed capital share is to posit that there is a large amount of “effective” capital $A_K K_t$. Given observed levels of capital K_t , this in turn implies that A_K must be high.

ment devoted to new automation, ϕ_t , equals

$$\phi_t = \frac{\dot{\alpha}_t \frac{K_t}{\alpha_t}}{\dot{K}_t} = \frac{1 - \alpha_t}{\alpha_t} \theta. \quad (\text{B-4})$$

Rearranging:

$$\frac{1 - \alpha_t}{\alpha_t} \theta = \phi_t g_K. \quad (\text{B-5})$$

Table 3 shows the results of inferring θ from (B-5) instead of from (B-2), for a range of values of the fraction of investment devoted to new automation. The conclusions are the same as those drawn from Figure 2 and Table 2.

α_t	$\phi_t = 5\%$		$\phi_t = 15\%$		$\phi_t = 25\%$		A_K
	θ	$\theta/(1 - \sigma)$	θ	$\theta/(1 - \sigma)$	θ	$\theta/(1 - \sigma)$	
0.1	0.01%	0.01%	0.02%	0.08%	0.03%	0.28%	0.33%
0.2	0.01%	0.03%	0.04%	0.19%	0.06%	0.62%	1.89%
0.3	0.02%	0.05%	0.06%	0.32%	0.11%	1.07%	5.20%
0.4	0.03%	0.08%	0.10%	0.50%	0.17%	1.66%	10.68%
0.5	0.05%	0.12%	0.15%	0.75%	0.25%	2.49%	18.66%
0.6	0.07%	0.19%	0.22%	1.12%	0.37%	3.73%	29.43%
0.7	0.12%	0.29%	0.35%	1.74%	0.58%	5.80%	43.27%
0.8	0.20%	0.50%	0.60%	2.98%	0.99%	9.95%	60.41%
0.9	0.45%	1.12%	1.34%	6.71%	2.24%	22.38%	81.10%

Table 3: $\frac{\theta}{1 - \sigma}$ and A_K as functions of the current level of automation, α_t , and the fraction of investment devoted to new automation (either 5%, 15%, or 25%). The automation rate θ is inferred from (B-5). The table uses $\sigma = 0.6$ throughout; adopting higher values of σ only strengthens the conclusion that (54) holds. Color shading is as in Table 2.

C Analysis of representative agent case

We consider the representative agent case, i.e., $\delta_o = \delta_w$. We simply write δ for this common value, and related, drop all group-specific subscripts.

By (16), capital and output asymptotically grow at the same rate. Moreover, consumption must asymptotically grow at this same rate, as follows. Certainly consumption cannot asymptotically

grow faster than F . Since F and K asymptotically grow at the same rate, this in turn implies that C cannot asymptotically grow faster than K . But nor can C asymptotically grow slower than K ; if it did, $\frac{C}{K} \rightarrow 0$, and so

$$g_K \rightarrow \frac{Y}{K} - \delta \geq F_K - \delta,$$

which would violate the transversality condition. Hence F , K , and C must all grow at the same rate asymptotically,

$$\lim g_Y = \lim g_K = \lim g_C. \tag{C-1}$$

In steady state, the representative agent both works and holds capital since $F_K = 0$ for $L = 0$ and vice versa, and so intra- and intertemporal optimality implies

$$g_C - g_{1-L} = \eta g_W, \tag{C-2}$$

$$\lim g_C = \eta (\bar{F}_K - \delta - \rho). \tag{C-3}$$

Lemma C-1 *In the representative agent benchmark, an equilibrium with a stable labor share exists if*

$$\theta < (1 - \sigma)(A_K - \delta - \rho). \tag{C-4}$$

The asymptotic growth rate of output, capital, and consumption is

$$\lim g_Y = \lim g_K = \lim g_C = \frac{\eta\theta}{1 - \sigma}. \tag{C-5}$$

Wages grow faster than consumption

$$\lim g_W = \frac{1}{\eta} \lim g_C, \tag{C-6}$$

while labor converges to 0 according to

$$\lim g_L = \left(1 - \frac{1}{\eta}\right) \lim g_C < 0. \tag{C-7}$$

The labor share converges to

$$\lim X = 1 - \left(\frac{\delta + \rho + \frac{\theta}{1-\sigma}}{A_K} \right)^{1-\sigma} \quad (\text{C-8})$$

Lemma C-2 *In the representative agent benchmark, an equilibrium with capital dominance exists if*

$$\theta > (1 - \sigma)(A_K - \delta - \rho). \quad (\text{C-9})$$

The asymptotic growth rate of output, capital, and consumption is

$$\lim g_Y = \lim g_K = \lim g_C = \eta(A_K - \delta - \rho). \quad (\text{C-10})$$

Wages grow faster than consumption,

$$\lim g_W = \frac{1}{\eta} \lim g_C, \quad (\text{C-11})$$

while labor converges to 0 according to

$$\lim g_L = \left(1 - \frac{\sigma}{\eta} \right) \lim g_C - \theta < 0. \quad (\text{C-12})$$

Proof of Lemma C-1: Recall that a stable labor share arises if wages asymptotically grow according to (14), and asymptotic capital and labor growth are linked via (A-3). From (14), wages grow without bound,

$$F_L \rightarrow \infty. \quad (\text{C-13})$$

Moreover, the asymptotic growth rate of leisure must be zero,

$$\lim g_{1-L} = 0, \quad (\text{C-14})$$

as follows. If instead $\lim g_{1-L} < 0$ then intratemporal optimality (C-2) and the complementarity of labor and leisure ($\eta < 1$) implies that $\lim g_W > \lim g_C$. But $\lim g_{1-L} < 0$ also implies that $\lim g_L = 0$, and hence (14) and (A-3) imply that $\lim g_W = g_K$, a contradiction (since $g_K = g_C$ by (C-1)).

So intratemporal optimality (C-2) implies that wages grow faster than consumption,

$$\lim g_W = \frac{1}{\eta} \lim g_C. \quad (\text{C-15})$$

Substituting in (14) gives

$$\lim g_C = \frac{\eta\theta}{1-\sigma}. \quad (\text{C-16})$$

Substituting into intertemporal optimality (C-3) gives

$$\bar{F}_K = \delta + \rho + \frac{\theta}{1-\sigma}. \quad (\text{C-17})$$

The condition for a stable labor share is simply $\bar{F}_K < A_K$, i.e.,

$$\theta < (1-\sigma)(A_K - \delta - \rho). \quad (\text{C-18})$$

The growth rate of labor is given by (A-3),

$$\lim g_L = \left(1 - \frac{1}{\eta}\right) g_C < 0. \quad (\text{C-19})$$

The consumption to capital ratio $\lim \frac{C}{K}$ is determined by the law of motion for capital: again using $g_C = g_K$ and intertemporal optimality (C-3),

$$\lim \frac{C}{K} = \frac{\bar{F}_K}{\lim \frac{KF_K}{F}} - \delta - g_K = \bar{F}_K^\sigma A_K^{1-\sigma} - \delta - \eta(\bar{F}_K - \delta - \rho), \quad (\text{C-20})$$

which is strictly positive since $\eta < 1$ and $A_K > \bar{F}_K > \delta$, completing the proof.

Proof of Lemma C-2: Under capital dominance, (18) holds. From (A-2), wages grow according to

$$\lim g_W = \frac{1}{\sigma} (\lim g_K - \lim g_L - \theta). \quad (\text{C-21})$$

The capital dominance condition is

$$\lim g_W < g_Y - \lim g_L. \quad (\text{C-22})$$

The asymptotic growth rate of leisure must be zero,

$$\lim g_{1-L} = 0, \tag{C-23}$$

as follows. The intratemporal optimality (C-2) condition implies $\lim g_C \leq \eta \lim g_W$. If $\lim g_{1-L} < 0$ then $\lim g_L = 0$, and the capital dominance condition reduces to $\lim g_W < \lim g_Y = \lim g_C$. Since $\eta < 1$, these two inequalities relating g_C to g_W contradict each other.

So intratemporal optimality (C-2) implies that wages grow faster than consumption,

$$\lim g_W = \frac{g_C}{\eta}. \tag{C-24}$$

In particular, (14) implies that wages grow without bound,

$$F_L \rightarrow \infty. \tag{C-25}$$

Substituting into intertemporal optimality (C-3) gives consumption growth in terms of the return on capital, which under complete automation is simply A_K :

$$\lim g_C = \eta(A_K - \delta - \rho). \tag{C-26}$$

Combining the two expressions above for the growth rate of wages, and using $g_K = g_C$, the growth rate of labor equals

$$\lim g_L = \left(1 - \frac{\sigma}{\eta}\right) g_C - \theta. \tag{C-27}$$

Note that the capital dominance condition and the expression for the growth rate of wages directly imply that

$$\lim g_L < 0.$$

The capital dominance condition rewrites as

$$\frac{g_C}{\eta} < \frac{\sigma}{\eta} g_C + \theta, \tag{C-28}$$

i.e.,

$$\theta > (1 - \sigma)(A_K - \delta - \rho). \quad (\text{C-29})$$

The consumption to capital ratio $\lim \frac{C}{K}$ is determined by the law of motion for capital,

$$\lim \frac{C}{K} = A_K - \delta - g_K, \quad (\text{C-30})$$

which is strictly positive since $\eta < 1$ and $A_K > \delta$, completing the proof.

D Analysis of $\eta = 1$ preferences

D.1 Flow utility

We first derive an expression for flow utility as $\eta \rightarrow 1$, using arguments standard to the analysis of CES production functions. We need to evaluate:

$$\lim_{\eta \rightarrow 1} \frac{1}{1-\gamma} \left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega (1-L_{i,t})^{\frac{\eta-1}{\eta}} \right)^{\frac{1-\gamma}{1-\frac{1}{\eta}}}.$$

Note that

$$\lim_{\eta \rightarrow 1} \left(C_{i,t}^{\frac{\eta-1}{\eta}} + \omega (1-L_{i,t})^{\frac{\eta-1}{\eta}} \right)^{\frac{1}{1-\frac{1}{\eta}}} = \lim_{\eta \rightarrow 1} \exp \left(\frac{1}{1-\frac{1}{\eta}} \ln \left(C_{i,t}^{1-\frac{1}{\eta}} + \omega (1-L_{i,t})^{1-\frac{1}{\eta}} \right) \right).$$

For notational convenience, write $x = 1 - \frac{1}{\eta}$. Recall that

$$\frac{\partial}{\partial x} z^x = \frac{\partial}{\partial x} e^{x \ln z} = (\ln z) z^x.$$

By l'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\ln \left(C_{i,t}^x + \omega (1-L_{i,t})^x \right)}{x} &= \lim_{x \rightarrow 0} \frac{\ln(C_{i,t}) C_{i,t}^x + \omega \ln(1-L_{i,t}) (1-L_{i,t})^x}{C_{i,t}^x + \omega (1-L_{i,t})^x} \\ &= \frac{\ln(C_{i,t}) + \omega \ln(1-L_{i,t})}{1+\omega} \\ &= \ln \left(C_{i,t}^{\frac{1}{1+\omega}} (1-L_{i,t})^{1-\frac{1}{1+\omega}} \right). \end{aligned}$$

To ease notation, define

$$\beta = \frac{1}{1+\omega}.$$

Hence flow utility is

$$\frac{1}{1-\gamma} \left(C_{i,t}^\beta (1-L_{i,t})^{1-\beta} \right)^{1-\gamma}. \tag{D-1}$$

D.2 Optimality conditions and preliminaries

From (D-1), the marginal utilities of consumption and leisure are

$$MU_{C_{i,t}} = \beta C_{i,t}^{\beta-1} (1 - L_{i,t})^{1-\beta} \left(C_{i,t}^\beta (1 - L_{i,t})^{1-\beta} \right)^{-\gamma} \quad (\text{D-2})$$

$$= \frac{\beta}{C_{i,t}} \left(C_{i,t}^\beta (1 - L_{i,t})^{1-\beta} \right)^{1-\gamma} \quad (\text{D-3})$$

$$MU_{1-L_{i,t}} = (1 - \beta) C_{i,t}^\beta (1 - L_{i,t})^{-\beta} \left(C_{i,t}^\beta (1 - L_{i,t})^{1-\beta} \right)^{-\gamma} \quad (\text{D-4})$$

$$= \frac{1 - \beta}{1 - L_{i,t}} \left(C_{i,t}^\beta (1 - L_{i,t})^{1-\beta} \right)^{1-\gamma} \quad (\text{D-5})$$

The intratemporal and intertemporal optimality conditions are

$$W_t \frac{\beta}{C_{i,t}} \leq \frac{1 - \beta}{1 - L_{i,t}}, \quad (\text{D-6})$$

$$\frac{\partial}{\partial t} \ln MU_{C_{i,t}} \leq -(R_t - \delta_i - \rho), \quad (\text{D-7})$$

with (D-6) at equality if labor is strictly positive ($L_{i,t} > 0$), and (D-7) at equality if capital-holding is strictly positive ($K_{i,t} > 0$).

Note that

$$\frac{1 - \beta}{\beta} = \omega,$$

so that (D-6) coincides with (21) evaluated at $\eta = 1$.

If type- i agents work then, from the intratemporal optimality condition (D-6),

$$g_{C_i} - g_{1-L_i} = g_W; \quad (\text{D-8})$$

and,

$$MU_{C,t} = \frac{\beta}{C_t} \left(C_t^\beta \left(\frac{\omega C_t}{W_t} \right)^{1-\beta} \right)^{1-\gamma} = \beta C_t^{-\gamma} \left(\frac{\omega}{W_t} \right)^{(1-\beta)(1-\gamma)}. \quad (\text{D-9})$$

Consequently, if type- i agents work their marginal utility grows according to

$$\frac{\partial}{\partial t} \ln MU_{C_i} = -\gamma g_{C_i} - (1 - \beta)(1 - \gamma) g_W$$

while if they are at the no-work corner,

$$\frac{\partial}{\partial t} \ln MU_{C_i} = (\beta(1-\gamma) - 1) g_{C_i} = -(1 - \beta + \beta\gamma) g_{C_i}. \quad (\text{D-10})$$

The above expressions all match the analogous expressions in the main text evaluated at $\eta = 1$.

The proof of Lemma 4 extends to $\eta = 1$, with just a couple of minor modifications.

Hence Corollary 1 continues to hold also.

Substituting $g_{1-L_i} = 0$ into (D-8) gives

$$g_{C_i} = g_W$$

for any group i that works. Hence regardless of whether group i works,

$$\frac{\partial}{\partial t} \ln MU_{C_i} = -(1 - \beta + \beta\gamma) g_{C_i}.$$

Part (I) of Proposition 1 continues to hold.

Part (II) of Proposition 1 continues to hold.

Part (III) of Proposition 1 continues to hold.

D.3 The intertemporal elasticity of substitution (IES)

For subsequent use, we calculate the intertemporal elasticity of substitution (IES) under asymptotically full automation. By definition,

$$IES_{i,t} = - \frac{\ln \frac{C_{i,t+1}}{C_{i,t}}}{\ln \frac{\frac{\beta}{C_{i,t+1}} (C_{i,t+1}^\beta (1-L_{i,t+1})^{1-\beta})^{1-\gamma}}{\frac{\beta}{C_{i,t}} (C_{i,t}^\beta (1-L_{i,t})^{1-\beta})^{1-\gamma}}}.$$

Since

$$\lim g_{1-L_i} = 0,$$

it follows that

$$\lim IES_i = \frac{1}{1 - \beta(1 - \gamma)}.$$

In contrast, for $\eta < 1$,

$$\lim IES_i = \eta.$$

D.4 Equilibrium characterization

Proposition D-1 *A capital-dominant equilibrium in which workers hold capital exists if*

$$\theta \geq \frac{1-\sigma}{1-\beta+\beta\gamma} (A_K - \delta_w - \rho) + \frac{\delta_w - \delta_o}{1-\beta+\beta\gamma}. \quad (\text{D-11})$$

Consumption growth of group i satisfies

$$\lim g_{C_i} = \frac{A_K - \delta_i - \rho}{1-\beta+\beta\gamma}. \quad (\text{D-12})$$

Labor converges to 0 according to

$$\lim g_{L_w} = \frac{1-\sigma}{1-\beta+\beta\gamma} (A_K - \delta_w - \rho) + \frac{\delta_w - \delta_o}{1-\beta+\beta\gamma} - \theta. \quad (\text{D-13})$$

Proposition D-2 *A capital-dominant equilibrium in which workers do not hold capital exists if*

$$\theta \in \left[\frac{1-\sigma}{1-\beta+\beta\gamma} (A_K - \delta_o - \rho), \frac{1-\sigma}{1-\beta+\beta\gamma} (A_K - \delta_w - \rho) + \frac{\delta_w - \delta_o}{1-\beta+\beta\gamma} \right]. \quad (\text{D-14})$$

Capitalists' consumption growth satisfies (D-12), while workers' consumption growth satisfies

$$\lim g_{C_w} = \frac{\lim g_{C_o} - \theta}{\sigma} < \lim g_{C_o}. \quad (\text{D-15})$$

Worker's labor is asymptotically constant

$$\lim g_{L_w} = 0. \quad (\text{D-16})$$

Proposition D-3 *A stable labor share equilibrium exists if*

$$\theta < \frac{1-\sigma}{1-\beta+\beta\gamma} (A_K - \delta_o - \rho). \quad (\text{D-17})$$

Consumption of capitalists and workers grows at same rate,

$$\lim g_{C_o} = \lim g_{C_w} = \frac{\theta}{1 - \sigma}. \quad (\text{D-18})$$

Worker's labor is asymptotically constant. The labor share converges towards

$$\lim X = 1 - \left(\frac{\delta_o + \rho + \frac{(1-\beta+\beta\gamma)\theta}{1-\sigma}}{A_K} \right)^{1-\sigma}. \quad (\text{D-19})$$

D.5 Equivalent characterization in terms of the IES

Using the characterization of the IES in subsection D.3, Propositions 2 - 4 for the case $\eta < 1$ and Propositions D-1 - D-3 for the case $\eta = 1$ can be written in unified manner to cover all $\eta \leq 1$, as follows:

Proposition D-4 *A capital-dominant equilibrium in which workers hold capital exists if*

$$\frac{\theta}{\lim IES} \geq \frac{1 - \sigma}{\eta} (A_K - \delta_w - \rho) + \delta_w - \delta_o. \quad (\text{D-20})$$

Consumption growth of group i satisfies

$$\frac{\lim g_{C_i}}{\lim IES} = A_K - \delta_i - \rho. \quad (\text{D-21})$$

Labor converges to 0 according to

$$\frac{\lim g_{L_w}}{\lim IES} = \left(1 - \frac{\sigma}{\eta} \right) (A_K - \delta_w - \rho) + \delta_w - \delta_o - \frac{\theta}{\lim IES}. \quad (\text{D-22})$$

Proposition D-5 *A capital-dominant equilibrium in which workers do not hold capital exists if*

$$\frac{\theta}{\lim IES} \in \left[\frac{1 - \sigma}{\eta} (A_K - \delta_o - \rho), \frac{1 - \sigma}{\eta} (A_K - \delta_w - \rho) + \delta_w - \delta_o \right]. \quad (\text{D-23})$$

Capitalists' consumption growth satisfies (D-21), while workers' consumption growth satisfies

$$\lim g_{C_w} = \eta \frac{\lim g_{C_o} - \theta}{\sigma + \eta - 1} < \lim g_{C_o}. \quad (\text{D-24})$$

Labor converges to 0 according to

$$\lim g_{L_w} = \frac{\eta - 1}{\eta} \lim g_{C_w}. \quad (\text{D-25})$$

Proposition D-6 *A stable labor share equilibrium exists if*

$$\frac{\theta}{\lim IES} < \frac{1 - \sigma}{\eta} (A_K - \delta_o - \rho). \quad (\text{D-26})$$

Consumption of capitalists and workers grows at same rate,

$$\lim g_{C_o} = \lim g_{C_w} = \frac{\eta \theta}{1 - \sigma}. \quad (\text{D-27})$$

Labor converges to 0 according to (D-25). The labor share converges to

$$\lim X = 1 - \left(\frac{\delta_o + \rho + \frac{\eta}{1 - \sigma} \frac{\theta}{IES}}{A_K} \right)^{1 - \sigma}. \quad (\text{D-28})$$

D.6 Proofs

Proof of Proposition D-1: We characterize the conditions for a capital-dominant equilibrium in which both groups hold capital to exist. From Proposition 1, workers work while capitalists do not. In a capital-dominant equilibrium, $\bar{F}_K = A_K$, and so from (28), the intertemporal conditions for capitalists and workers are

$$\begin{aligned} \lim g_{C_o} &= \frac{A_K - \delta_o - \rho}{1 - \beta + \beta\gamma} \\ \lim g_{C_w} &= \frac{A_K - \delta_w - \rho}{1 - \beta + \beta\gamma} \end{aligned}$$

while the intratemporal condition for workers is (using Lemma 4)

$$\lim g_W = \lim g_{C_w} = \frac{A_K - \delta_w - \rho}{1 - \beta + \beta\gamma}.$$

(Note that the above expression is positive by assumption (8).) Capital holdings grow according to

$$\begin{aligned}\lim g_{K_o} &= A_K - \delta_o - \lim \frac{C_o}{K_o} \\ \lim g_{K_w} &= A_K - \delta_w + \lim \frac{L_w F_L - C_w}{K_w},\end{aligned}$$

and from (A-2), wages grow according to

$$\lim g_W = \frac{1}{\sigma} (\lim g_K - \lim g_{L_w} - \theta).$$

Capitalists' transversality condition implies that C_o and K_o asymptotically grow at the same rate:

$$\lim g_{K_o} = \lim g_{C_o} = \frac{A_K - \delta_o - \rho}{1 - \beta + \beta\gamma}.$$

We characterize an equilibrium in which C_w and K_w asymptotically grow at the same rate. In this case,

$$\lim g_{K_w} < \lim g_{K_o} = \lim g_K,$$

and so

$$\lim g_{L_w} = \frac{A_K - \delta_o - \rho}{1 - \beta + \beta\gamma} - \sigma \frac{A_K - \delta_w - \rho}{1 - \beta + \beta\gamma} - \theta.$$

A worker's transversality condition is equivalent to

$$\lim g_{C_w} \geq \lim g_W + \lim g_{L_w}, \tag{D-29}$$

which substituting in the above expressions is equivalent to

$$0 \geq \frac{A_K - \delta_o - \rho}{1 - \beta + \beta\gamma} - \sigma \frac{A_K - \delta_w - \rho}{1 - \beta + \beta\gamma} - \theta,$$

and hence to

$$\theta \geq \frac{1 - \sigma}{1 - \beta + \beta\gamma} (A_K - \delta_o - \rho) + \frac{\sigma (\delta_w - \delta_o)}{1 - \beta + \beta\gamma} = \frac{1 - \sigma}{1 - \beta + \beta\gamma} (A_K - \delta_w - \rho) + \frac{\delta_w - \delta_o}{1 - \beta + \beta\gamma}. \quad (\text{D-30})$$

Note that $\lim g_{C_o} > \lim g_{C_w}$ together with the worker transversality condition (D-29) implies that the capital-dominance condition is satisfied; and also that capitalists indeed do not work. Moreover, the worker transversality condition implies that $\lim g_{L_w} < 0$.

Proof of Proposition D-2: We characterize the conditions for a capital-dominant equilibrium in which workers do not hold capital to exist. By the similar arguments to those in the proof of Proposition 2, the asymptotic equilibrium conditions are as follows. (Relative to the proof of Lemma 2, the key difference is that workers' intertemporal optimality condition is replaced with an intratemporal budget constraint.)

$$\begin{aligned} \lim g_{K_o} = \lim g_{C_o} &= \frac{A_K - \delta_o - \rho}{1 - \beta + \beta\gamma} \\ \lim g_W &= \lim g_{C_w} \\ \lim g_{C_w} &= \lim g_W + \lim g_{L_w} \\ \lim g_W &= \frac{1}{\sigma} (\lim g_{K_o} - \lim g_{L_w} - \theta). \end{aligned}$$

From a worker's intratemporal optimality and intratemporal budget constraint,

$$\lim g_{L_w} = 0.$$

Hence

$$\lim g_W = \frac{\lim g_{K_o} - \theta}{\sigma}.$$

The capital-dominance condition is $\lim g_{K_o} > \lim g_W + \lim g_{L_w}$. Note that if the capital-dominance condition holds then $\lim g_{C_o} > \lim g_{C_w}$, which ensures that capitalists indeed do not work asymptotically. Substituting in, the capital-dominance condition is

$$\lim g_{K_o} > \frac{\lim g_{K_o} - \theta}{\sigma}.$$

The condition that workers asymptotically do not want to hold capital is (from (28), and substituting in for $\lim g_{C_w}$)

$$\lim g_W \geq \frac{A_K - \delta_w - \rho}{1 - \beta + \beta\gamma},$$

i.e.,

$$\lim g_W = \frac{\lim g_{K_o} - \theta}{\sigma} \geq \frac{A_K - \delta_w - \rho}{1 - \beta + \beta\gamma} = \lim g_{K_o} - \frac{\delta_w - \delta_o}{1 - \beta + \beta\gamma}.$$

The above condition and (8) imply that $\lim g_W > 0$.

Hence an equilibrium of this type exists if either $\sigma + \eta > 1$ and

$$\theta \in \left[(1 - \sigma) \lim g_{K_o}, (1 - \sigma) \lim g_{K_o} + \sigma \frac{\delta_w - \delta_o}{1 - \beta + \beta\gamma} \right].$$

Substituting in for $\lim g_{K_o}$ yields the result.

Proof of Proposition D-3: We characterize the conditions for a stable labor share equilibrium to exist. From Proposition 1, workers do not hold capital. Following similar steps to those in the proofs of Propositions 2 and 3, but incorporating the possibility that capitalists work, the asymptotic equilibrium conditions are

$$\begin{aligned} \lim g_{C_o} &\geq \lim g_W \\ \lim g_{C_o} &= \frac{\bar{F}_K - \delta_o - \rho}{1 - \beta + \beta\gamma} \\ \lim g_{K_o} &= \bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o} \\ \lim g_W &= \lim g_{C_w} \\ \lim g_{C_w} &= \lim g_W + \lim g_{L_w} \\ \lim g_W &= \frac{\theta}{1 - \sigma}. \end{aligned}$$

From Part (III) of Proposition 1,

$$\lim g_Y = \lim g_{K_o} = \lim g_{C_o} = \frac{\bar{F}_K - \delta_o - \rho}{1 - \beta + \beta\gamma}.$$

We first show that aggregate labor growth matches worker-labor growth, i.e.,

$$\lim g_L = \lim g_{L_w}. \quad (\text{D-31})$$

If capitalists do not work then (D-31) immediate. If instead capitalists work, note that capital evolves according to

$$\lim g_{K_o} = \bar{F}_K - \delta_o - \lim \frac{C_o - F_L L_o}{K_o}.$$

Capitalists' transversality constraint implies that their labor income grows weakly slower than the common growth rate of their consumption and capital. Moreover, if both capitalists and workers work, their consumption growth rates must asymptotically coincide (by Lemma 4 and the intratemporal optimality conditions). Hence

$$\lim g_W + \lim g_{L_o} \leq \lim g_{C_o} = \lim g_{C_w} = \lim g_W + g_{L_w}, \quad (\text{D-32})$$

implying that $\lim g_{L_o} \leq \lim g_{L_w}$ and establishing (D-31).

From the workers' intratemporal optimality and intratemporal budget constraint,

$$\lim g_{L_w} = 0.$$

Further, from (A-3), a stable labor share requires

$$\lim g_{K_o} - \lim g_L = \frac{\theta}{1 - \sigma}.$$

From (D-31), it follows that

$$\lim g_{K_o} = \frac{\theta}{1 - \sigma},$$

which combined with capitalists' intertemporal optimality implies that the limiting rental rate is

$$\bar{F}_K = \frac{(1 - \beta + \beta\gamma)}{1 - \sigma} \theta + \delta_o + \rho. \quad (\text{D-33})$$

From (17), the asymptotic capital share is bounded away from one if and only if $\left(\frac{\bar{F}_K}{A_K}\right)^{1-\sigma} < 1$,

which after substitution for \bar{F}_K is equivalent to

$$\frac{(1 - \beta + \beta\gamma)}{1 - \sigma} \theta + \delta_o + \rho < A_K.$$

Rearranging establishes the stable labor share condition, (35).

Workers' and capitalists' consumption grow at the same asymptotic rate, as follows. If capitalists do not work, this is immediate from the combination of definition of a stable labor share and the fact that output F , capital K_o and capitalist consumption C_o all grow at the same rate. If instead capitalists work, then it follows intratemporal optimality conditions, as already noted in (D-32).

Finally, the expression for the limiting labor share follows from the substitution of \bar{F}_K into (17). This completes the proof.

E Labor-share evolution away from steady state

From (13),

$$g_{X,t} = (1 - \sigma) g_{W,t} - \theta.$$

From (A-2),

$$g_{W,t} = \frac{1}{\sigma} \left((1 - X_t) (g_{K,t} - g_{L,t}) + \frac{\theta}{1 - \sigma} \left(\sigma - \frac{1 - X_t}{\alpha_t} \right) \right).$$

Hence

$$\begin{aligned} g_{X,t} &= \frac{1 - \sigma}{\sigma} \left((1 - X_t) (g_{K,t} - g_{L,t}) + \frac{\theta}{1 - \sigma} \left(\sigma - \frac{1 - X_t}{\alpha_t} \right) \right) - \theta \\ &= \frac{1 - \sigma}{\sigma} \left((1 - X_t) (g_{K,t} - g_{L,t}) - \frac{\theta}{1 - \sigma} \frac{1 - X_t}{\alpha_t} \right) \\ &= \frac{1 - X_t}{\sigma} \left((1 - \sigma) (g_{K,t} - g_{L,t}) - \frac{\theta}{\alpha_t} \right). \end{aligned}$$

The empirical value of $g_{K,t} - g_{L,t}$ was approximately 1 percentage point higher in the 1950-1970 period than subsequently. So very roughly, the observed trends of K and L predict that labor share growth should be higher in the earlier period by approximately

$$0.3 \frac{.4}{.6} 1\% = 0.2\%.$$

The estimated value of g_X in the later period, which we use in our calibration, is $g_X = -0.17\%$. So a flat labor share prior to 1970 is consistent with these calculations. (Note: These calculations don't incorporate the increase in automation α_t , which would lead to an upwards shift to g_X .)

F Proof of Lemma 3

In this appendix, we write the dynamic equilibrium as a vector ODE in the state variables (K_o, K_w, C_o, C_w) .

We proceed in five steps.

Step 1: $L_i, R, W,$ and X as functions of the state. Given state variables

$$(K_o, K_w, C_o, C_w),$$

we can solve for L_o and L_w . Intratemporal optimality is

$$WC_i^{-\frac{1}{\eta}} = \omega(1 - L_i)^{-\frac{1}{\eta}},$$

i.e.,

$$L_i = 1 - \omega^\eta W^{-\eta} C_i.$$

Note that L_i is increasing in wages W . Wages W are decreasing in each of L_o, L_w . Hence L_i is decreasing in L_{-i} (for $i = o, w$), and accordingly, L_o and L_w (and, hence, also W) are uniquely determined by (K_o, K_w, C_o, C_w) . The same follows for R given L .

Step 2: Evolution of capital. Capital evolves straightforwardly according to

$$g_{K_i} = R + W \frac{L_i}{K_i} - \delta_i - \frac{C_i}{K_i} \tag{F-1}$$

where aggregate capital growth is defined as

$$g_K = \frac{K_o g_{K_o} + K_w g_{K_w}}{K_o + K_w}. \tag{F-2}$$

and R, W

Step 3: Evolution of wages given g_{C_o} and g_{C_w} . From (11) the production function implies

$$\begin{aligned} g_W &= \frac{1}{\sigma} (g_Y - g_L - \theta) \\ &= \frac{1}{\sigma} \left((1-X)(g_K - g_L) + \frac{\theta}{1-\sigma} \left(1 - \frac{1-X}{\alpha} \right) - \theta \right) \end{aligned} \quad (\text{F-3})$$

where

$$g_L = \frac{\lambda_o L_o}{\lambda_o L_o + \lambda_w L_w} g_{L_o} + \frac{\lambda_w L_w}{\lambda_o L_o + \lambda_w L_w} g_{L_w}.$$

For a household-type i such that

$$C_i < \left(\frac{W}{\omega} \right)^\eta,$$

labor supply $L_i > 0$ and intratemporal optimality (21) implies

$$\eta g_W = g_{C_i} + g_{L_i} \frac{L_i}{1-L_i}.$$

Consider first the case in which only one group $i \in \{o, w\}$ works

$$\frac{1-L_i}{L_i} \eta g_W = \frac{1-L_i}{L_i} g_{C_i} + g_L.$$

Plugging this into (F-3) yields

$$\begin{aligned} g_W + \frac{1-X}{\sigma} g_L &= g_W + \frac{1-X}{\sigma} \left(\frac{1-L_i}{L_i} \eta g_W - \frac{1-L_i}{L_i} g_{C_i} \right) \\ &= \frac{1}{\sigma} \left((1-X) g_K + \frac{\theta}{1-\sigma} \left(1 - \frac{1-X}{\alpha} \right) - \theta \right) \end{aligned}$$

which delivers the following expression for wage growth.

$$g_W = \frac{(1-X) \frac{1-L_i}{L_i} g_{C_i} + (1-X) g_K + \frac{\theta}{1-\sigma} \left(1 - \frac{1-X}{\alpha} \right) - \theta}{\sigma + \eta (1-X) \frac{1-L_i}{L_i}}. \quad (\text{F-4})$$

Next, consider the case in which both groups $i \in \{o, w\}$ work. Combining the intratemporal

optimality conditions yields

$$\left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} \right) \eta g_W = \left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} g_{C_o} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} g_{C_w} \right) + g_L.$$

In parallel to the above, this yields the following expression for wage growth:

$$\begin{aligned} & g_W + \frac{1-X}{\sigma} g_L \\ = & g_W + \frac{1-X}{\sigma} \left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} \right) \eta g_W \\ & - \frac{1-X}{\sigma} \left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} g_{C_o} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} g_{C_w} \right) \\ = & \frac{1}{\sigma} \left((1-X) g_K + \frac{\theta}{1-\sigma} \left(1 - \frac{1-X}{\alpha} \right) - \theta \right) \end{aligned}$$

and so

$$g_W = \frac{\left((1-X) g_K + \frac{\theta}{1-\sigma} \left(1 - \frac{1-X}{\alpha} \right) - \theta \right) + (1-X) \left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} g_{C_o} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} g_{C_w} \right)}{\sigma + \eta(1-X) \left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} \right)}. \quad (\text{F-5})$$

The cases combine to yield:

$$g_W = \frac{\left((1-X) g_K + \frac{\theta}{1-\sigma} \left(1 - \frac{1-X}{\alpha} \right) - \theta \right) + (1-X) \left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} g_{C_o} \mathbf{1}_{L_o > 0} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} g_{C_w} \mathbf{1}_{L_w > 0} \right)}{\sigma + \eta(1-X) \left(\frac{\lambda_o(1-L_o)}{\lambda_o L_o + \lambda_w L_w} \mathbf{1}_{L_o > 0} + \frac{\lambda_w(1-L_w)}{\lambda_o L_o + \lambda_w L_w} \mathbf{1}_{L_w > 0} \right)}. \quad (\text{F-6})$$

Step 4: Evolution of consumption. Consider first the case in which group i households work but do not hold capital. Their budget constraint implies

$$g_{C_i} = g_{L_i} + g_W = \frac{1-L_i}{L_i} (\eta g_W - g_{C_i}) + g_W.$$

Rearranging yields

$$g_{C_i} = ((1-L_i)\eta + L_i) g_W.$$

If, instead, $K_i > 0$, consumption growth follows the Euler equation (22), which combines with the expressions for marginal utility growth (24) and (25), depending on whether i -households work

or not. Combining all cases.

$$\begin{aligned}
g_{C_i} &= \left([((1 - L_i)\eta + L_i)g_W] \mathbf{1}_{K_i=0} + \left[\frac{1}{\gamma} (R - \delta_i - \rho) - \frac{1 - \eta\gamma}{\gamma} \frac{\omega^\eta g_W}{W^{\eta-1} + \omega^\eta} \right] \mathbf{1}_{K_i>0} \right) \mathbf{1}_{L_i>0} \\
&+ \frac{C_i^{1-\frac{1}{\eta}} + \omega}{\gamma C_i^{1-\frac{1}{\eta}} + \frac{\omega}{\eta}} (R - \delta_i - \rho) \mathbf{1}_{L_i=0}. \tag{F-7}
\end{aligned}$$

Step 5: Combining. Substituting in (F-6), equations (F-1) and (F-7) give a first-order vector ODE.

G Labor-augmenting technological change (growth in A_L)

Here, we consider the generalization of our analysis to the case in which labor productivity A_L grows over time. We show that the automation rate θ is replaced by an “augmented” automation rate $\theta + (1 - \sigma)g_{A_L}$ in many key equations, including in particular equation (54) that gives the threshold for a stable labor share equilibrium, and equation (57) used to estimate automation rate.

Specifically: Under this generalization, wages and the labor share evolve according to

$$g_{W,t} = \frac{1}{\sigma} (g_{Y,t} - g_{L,t} - (\theta + (1 - \sigma)g_{A_L})) \quad (\text{G-1})$$

$$g_{X,t} = (1 - \sigma)g_{W,t} - (\theta + (1 - \sigma)g_{A_L}). \quad (\text{G-2})$$

Hence a stable labor share requires

$$\lim g_W = \frac{\theta + (1 - \sigma)g_{A_L}}{1 - \sigma}. \quad (\text{G-3})$$

As in the baseline case of $g_{A_L} = 0$, for workers to continue to work wage growth must satisfy

$$\eta \lim g_W = \lim g_{C_w}, \quad (\text{G-4})$$

while worker and capitalist transversality conditions in a stable labor equilibrium, and capitalist intertemporal optimality, imply

$$\lim g_{C_w} = \lim g_{C_o} = \eta (\bar{F}_K - \delta_o - \rho). \quad (\text{G-5})$$

Hence a stable labor share equilibrium requires

$$\frac{\theta + (1 - \sigma)g_{A_L}}{1 - \sigma} = \bar{F}_K - \delta_o - \rho, \quad (\text{G-6})$$

which in turn requires

$$\frac{\theta + (1 - \sigma)g_{A_L}}{1 - \sigma} \leq A_K - \delta_o - \rho, \quad (\text{G-7})$$

thereby generalizing the threshold condition (54).

Parallel to the baseline case of $g_{A_L} = 0$, substitution of (G-1) into (G-2) yields

$$\theta + (1 - \sigma) g_{A_L} = (1 - \sigma) (g_{Y,t} - g_{L,t}) - \sigma g_{X,t}. \quad (\text{G-8})$$

Consequently: the RHS of (G-8) gives an estimate of $\theta + (1 - \sigma) g_{A_L}$, which is exactly the quantity that enters the generalized threshold condition (G-7).